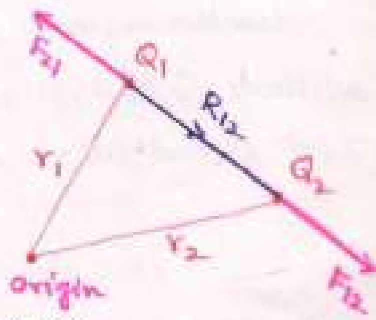
UNIT-I : ELECTROSTATICS

~~Gota~~

Coulomb's Law: In 1785, Charles Augustin de Coulomb, a colonel in the French Army performed an elaborate series of experiments using a delicate torsion balance to determine quantitatively the force exerted between two objects, each having a static charge of electricity. He published results bears a great similarity to Newton's gravitational law. Coulomb stated that the force between two very small objects separated in a vacuum or free space by a distance which is large compared to their size is proportional to the charge on each and inversely proportional to the square of the distance between them, or

$$F = K\frac{Q_1Q_2}{R^2} \longrightarrow ①$$



Statement: The force F between two point charges Q_1 and Q_2 is

1. Along the line joining them
2. Directly proportional to the product Q_1Q_2 of the charges
3. Inversely proportional to the square of the distance R between them.

where K is the proportionality constant whose value depends on the choice of the system of units.

In SI units Q_1 & Q_2 are in coulombs (C), R in meters & force in New

and $K = 1/4\pi\epsilon_0$ $\quad \epsilon_0 = 8.854\times10^{-12} \simeq \frac{10^{9}}{36\pi}$ F/m

$$\therefore K = \frac{1}{4\pi\epsilon_0} = 9\times10^{9}\ m/F$$

$\therefore$ Eq ① becomes

$$\boxed{F = \frac{Q_1Q_2}{4\pi\epsilon_0R^2}}$$

Vector form

If point charges Q_1 and Q_2 are located at points having positio vectors $\bar{r}_1$ and $\bar{r}_2$ then the force F_{12} due on Q_2 due to Q_1 is given

$$\boxed{\overline{F_{12}} = \frac{Q_1Q_2}{4\pi\epsilon_0R^2}\,\bar{a}_{R_{12}}} \longrightarrow ②$$

$$\text{where} \quad \left.\begin{aligned} \overline{R_{12}} &= \bar{r}_2 - \bar{r}_1 \\ R &= |\overline{R_{12}}| \\ \bar{a}_{R_{12}} &= \frac{\overline{R_{12}}}{R} \end{aligned}\right\} \longrightarrow ③$$

Substituting eq③ into eq②, we may write eq② as

$$\boxed{\begin{aligned}\overline{F_{12}} &= \frac{Q_1 Q_2}{4\pi\epsilon_0 R^3}\overline{R_{12}} \\ \overline{F_{12}} &= \frac{Q_1 Q_2}{4\pi\epsilon_0}\frac{(\bar r_2 - \bar r_1)}{|\bar r_2 - \bar r_1|^3}\end{aligned}}$$

$$\begin{aligned}F_{21} &= \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{21}^2}\bar a_{21} \\ &= \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{21}^2}(-\bar a_{12}) \\ &= -F_{12}\end{aligned}$$

Note:

1. The force F_{21} on Q_1 due to Q_2 is given by

$$\overline{F_{21}} = |\overline{F_{12}}|\,\bar a_{R_{21}} = |\overline{F_{12}}|(-\bar a_{R_{12}})$$

or $$F_{21} = -F_{12}$$

since $$\bar a_{R_{21}} = -\bar a_{R_{12}}$$

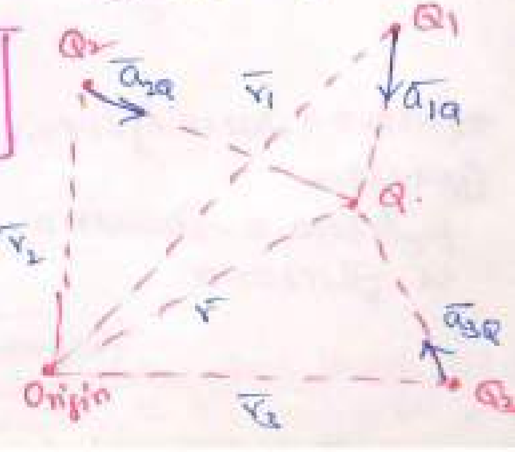
2. Like charges repel each other, while unlike charges attract.
3. The distance R between the charged bodies Q_1 and Q_2 must be large compared with the linear dimensions of the bodies; that i Q and Q_2 must be point charges.
4. Q_1 and Q_2 must be static
5. The sign of Q_1 and Q_2 must be taken into account in eq② for like charges $QQ_2 > 0$. For unlike charges $Q_1Q_2 < 0$

If we have more than two point charges, we can used the principle of super-position to determine the force on a particular charge. The principle states that if there are N charges $Q_1, Q_2, Q_3 \ldots Q_N$ located, respectively, at points with position vectors $\bar r_1, \bar r_2, \ldots \bar r_N$, the resultant force F on the charge Q located at point $\bar r$ is the vector sum of the forces exerted on Q by each of the charges $Q_1, Q_2, \ldots Q_N$. Hence:

$$F = \frac{QQ_1(\bar r - \bar r_1)}{4\pi\epsilon_0|\bar r - \bar r_1|^3} + \frac{QQ_2(\bar r - \bar r_2)}{4\pi\epsilon_0|\bar r - \bar r_2|^3} + \cdots + \frac{QQ_N(\bar r - \bar r_N)}{4\pi\epsilon_0(\bar r - \bar r_N)^3}$$

or

$$\boxed{F = \frac{Q}{4\pi\epsilon_0}\sum_{k=1}^{N}\frac{Q_k(\bar r - \bar r_k)}{|\bar r - \bar r_k|^3}}$$

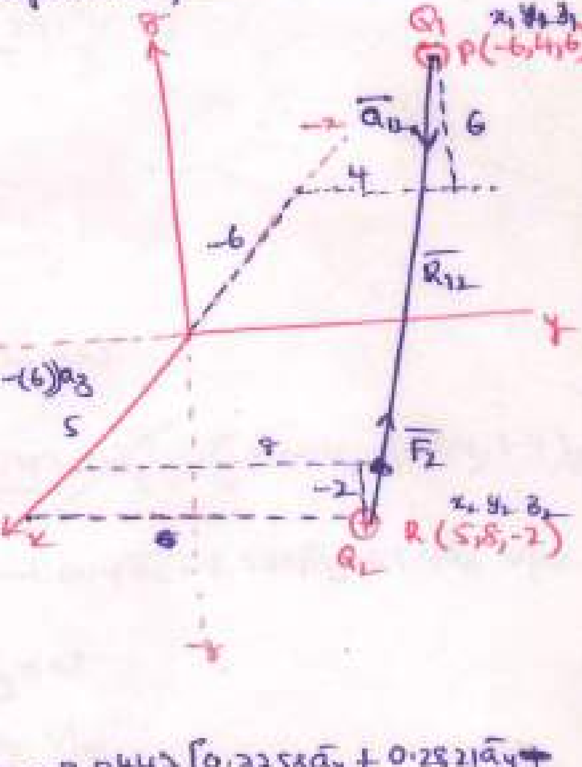


Steps to Solve Problems on Coulomb's Law:

1. Obtain the position vectors of the points where the charges are located.
2. Obtain the unit vectors along the straight line joining the charges. The direction is towards the charge on which the force exerted is to be calculated.
3. Using Coulomb's law, express the force exerted in the vector form.
4. If there are more charges, repeat steps 1 to 3 for each charge exerting a force on the charge under consideration.
5. Using the principle of superposition, the vector sum of all the forces calculated earlier is the resultant force, exerted on the charge under consideration.

<u>Example 1.</u> A charge $Q_1 = -20\mu C$ is located at $P(-6,4,6)$ and a charge $Q_2 = 50\mu C$ is located at $R(5,8,-2)$ in a free space. Find the force exerted on Q_2 by Q_1 in vector form. The distance given are in meters!

<u>Solution</u>: From the coordinates the respective position vectors are



$$\bar{P} = -6\bar{a}_x + 4\bar{a}_y + 6\bar{a}_z$$

and $\bar{R} = 5\bar{a}_x + 8\bar{a}_y - 2\bar{a}_z$

The force on Q_2 is given by

$$\bar{F}_{21} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\bar{a}_{12}$$

$$\bar{R}_{12} = \bar{R}_{PR} = \bar{R} - \bar{P} = (5-(-6))\bar{a}_x + (8-4)\bar{a}_y + (-2-(6))\bar{a}_z$$

$$= 11\bar{a}_x + 4\bar{a}_y - 8\bar{a}_z$$

$$|\bar{R}_{12}| = \sqrt{11^2 + 4^2 + (8)^2} = 14.1774$$

$$\bar{a}_{12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{11\bar{a}_x + 4\bar{a}_y + -8\bar{a}_z}{14.1774}$$

$$\bar{a}_{12} = 0.7758\bar{a}_x + 0.2821\bar{a}_y + 0.5642\bar{a}_z$$

$$F_{21} = \frac{-20\times10^{-6}\times50\times10^{-6}}{4\pi\times8.854\times10^{-12}\times(14.1774)^2}[\bar{a}_{12}] = -0.0443[0.7758\bar{a}_x + 0.2821\bar{a}_y + 0.5642 a_z]$$

$$= -0.0346\bar{a}_x - 0.01261\bar{a}_y + 0.02522\bar{a}_z \text{ N}$$

The magnitude

$$\bar{F}_{21} = \sqrt{(0.0346)^2 + (0.01241)^2 + (-0.02522)^2}$$

$$= \underline{44.634 \text{ mN}}.$$

F_2 is an attractive force, it acts in opposite direction to $\bar{a}_{12}$ which is indicated by -ve sign in an

Alternative method.

Example: point charges 5nC and −2nC are located at (2,0,4) and (−3,0,5) respectively.

(a) Determine the force on a 1nC point charge at (1,−3,7)

(b) Find Electric Field Intensity $\bar{E}$ at (1,−3,7)

Solution:

$$\bar{F} = \sum_{k=1,2} \frac{QQ_k}{4\pi\epsilon_0 R^2}\bar{a}_R = \sum_{k=1,2} \frac{QQ_k(\bar{r}-\bar{r}_k)}{4\pi\epsilon_0 |\bar{r}-\bar{r}_k|^3}$$

$$= \frac{Q}{4\pi\epsilon_0}\left\{\frac{5\times10^{-9}[(1,-3,7)-(2,0,4)]}{|(1,-3,7)-(3,2,-1)|^3} + \frac{-2\times10^{-9}[(1,-3,7)-(-3,0,5)]}{|(1,-3,7)-(-3,0,5)|^3}\right.$$

$$= \frac{10^{-9}\times10^{-9}}{4\pi\epsilon_0}\left\{\frac{5(-1,-3,3)}{[(-2)^2+(-5)^2+(8)^2]^{3/2}} - \frac{2(4,-3,2)}{(4^2+(-3)^2+2^2)^{3/2}}\right\}$$

$$= \frac{10^{-9}\times\cancel{10^{-9}}}{4\pi\times\frac{\cancel{10^{-9}}}{36\pi}}\left\{\frac{(-5,-15,15)}{(4+25+64)^{3/2}} - \frac{(8,-6,4)}{(16+9+4)^{3/2}}\right\}$$

$$= \frac{\overset{9}{\cancel{36}}\times10^{-9}}{4}\left\{\frac{(-5,-15,15)}{(93)^{3/2}} - \frac{(8,-6,4)}{(29)^{3/2}}\right\}$$

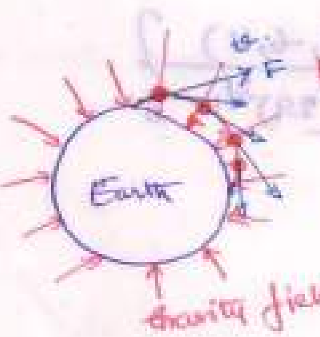
$$\bar{F} = -1.004\bar{a}_x - 1.284\bar{a}_y + 1.4\bar{a}_z \text{ nN}$$

$$\bar{E} = \frac{\bar{F}}{Q} = \frac{(-1.004, -1.284, 1.4)\times\cancel{10^{-9}}}{\cancel{10^{-9}}} = (-1.004, -1.284, 1.4)\bar{a}$$

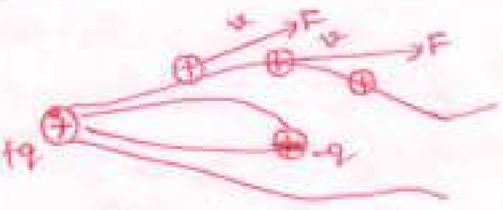
$$= -1.004\bar{a}_x - 1.284\bar{a}_y + 1.4\bar{a}_z \text{ V/m.}$$

$$\bar{F} = -1.004\bar{a}_x - 1.284\bar{a}_y + 1.4\bar{a}_z \text{ nN}$$

$$\bar{E} = -1.004\bar{a}_x - 1.284\bar{a}_y + 1.4\,a_z \text{ V/m.}$$



An object moving with a velocity in the vicinity of Earth's Gravitational field.



+ve charge moving with a vel
in the vicinity of Electric fie

A point charge $Q_A = 1\mu c$ is at $A(0,0,1)$ and $Q_B = -1\mu c$ at $B(0,0,-1)$. Find E_r, E_θ, and E_ϕ at $P(1,2,3)$.

The vector joining $A(0,0,1)$ ~~and~~ towards $P(1,2,3)$ is

$$\bar{r}_{AP} = \bar{a}_x + 2\bar{a}_y + 3\bar{a}_z \cancel{-\bar{a}_z} - 0\bar{a}_y - 1\bar{a}_z$$

$$= \bar{a}_x + 2\bar{a}_y + 2\bar{a}_z$$

Its magnitude is towards $|\bar{r}_{AP}| = \sqrt{1^2+2^2+2^2} = 3$

Similarly vector joining point $B(0,0,-1)$ towards $P(1,2,3)$ is

$$\bar{r}_{BP} = \bar{a}_x + 2\bar{a}_y + 3\bar{a}_z - 0\bar{a}_x - 0\bar{a}_y - (-\bar{a}_z)$$

$$= \bar{a}_x + 2\bar{a}_y + 4\bar{a}_z$$

Its magnitude is $|\bar{r}_{BP}| = \sqrt{1^2+2^2+4^2} = \sqrt{21}$

Thus Electric field in Cartesian coordinate system at point P due to the given charges is

$$E = \frac{Q_A}{4\pi\epsilon_0 |\bar{r}_{AP}|^2} \cdot \frac{\bar{r}_{AP}}{|\bar{r}_{AP}|} + \frac{Q_B}{4\pi\epsilon |\bar{r}_{BP}|^2} \frac{\bar{r}_{BP}}{|\bar{r}_{BP}|}$$

$$= \frac{10^{-6}}{4\pi\epsilon_0 \times 3^2} \frac{(\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z)}{3} - \frac{10^{-6}}{4\pi\epsilon_0 \times 21} \cdot \frac{\bar{a}_x + 2\bar{a}_y + 4\bar{a}_z}{\sqrt{21}}$$

$$= 333.2(\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z) - 93.5(\bar{a}_x + 2\bar{a}_y + 4\bar{a}_z)$$

$$E = 239.7\,\bar{a}_x + 479.4\,\bar{a}_y + 292.4\,\bar{a}_z \quad V/m.$$

$$E = E_x\bar{a}_x + E_y\bar{a}_y + E_z\bar{a}_z$$

The conversion of point $P(1,2,3)$ in spherical coordinate system ~~[illegible]~~ (r,θ,ϕ)

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}} = \cos^{-1}\frac{3}{\sqrt{14}} = 36.7^\circ$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{1} = 63.43^\circ$$

~~[illegible]~~

$\bar{a}_r = \sin\theta\cos\phi\,\bar{a}_x + \sin\theta\sin\phi\,\bar{a}_y + \cos\theta\,\bar{a}_z$

$\bar{a}_\theta = \cos\theta\cos\phi\,\bar{a}_x + \cos\theta\sin\phi\,\bar{a}_y \cancel{+} - \sin\theta\,\bar{a}_z$

$\bar{a}_\phi = -\sin\phi\,\bar{a}_x + \cos\phi\,\bar{a}_y$

The components of electric field Intensity in the spherical coordinate system are

$$\boxed{E_r = \underline{E_x}\sin\theta\cos\phi + \underline{E_y}\sin\theta\sin\phi + \underline{E_z}\cos\theta}$$

$$= 239.7\sin 36.7^\circ\cos 63.43^\circ + 479.4\sin 36.7^\circ \cancel{\cos}\,63.43^\circ + 292.4\cos 36.7^\circ$$

$$= 554.7\,V/m$$

Similarly

$E_\theta = E_x\cos\theta\cos\phi + E_y\cos\theta\sin\phi - E_z\sin\theta. = 255.\,V/m$

$E_\phi = -E_x\sin\phi + E_y\cos\phi = 0.05 \approx 0\,V/m.$

∴ EFI in Spherical coordinate system $\bar{E}_s = 554.7\,\bar{a}_r + 255\bar{a}_\theta$ V/m

A 20nC point charge is located at P(2,4,-3) in free space (a) Find $\bar{E}(r)$ (b) Find $\bar{E}$ at A(-3,2,0) (c) Find the locus of all points at which $E_x = 1$ V/m

The given point charge is Q = 20nC at P(2,4,-3)

(a) The vector joining point P(2,4,3) towards any point M(x,y,z) is

$$\bar{r}_{pm} = (x-2)\bar{a}_x + (y-4)\bar{a}_y + (z+3)\bar{a}_z$$

The magnitude is $|r_{pm}| = \sqrt{(x-2)^2+(y-4)^2+(z+3)^2}$

∴ EFI $\bar{E}(r)$ in Cartesian coordinate system at point P due to the given charge is

$$\bar{E}(r) = \frac{Q}{4\pi\epsilon_0 |\bar{r}_{pm}|^2} \frac{\bar{r}_{pm}}{|r_{pm}|}$$

$$= \frac{20\times10^{-9}}{4\pi\epsilon_0} \frac{(x-2)\bar{a}_x+(y-4)\bar{a}_y+(z+3)\bar{a}_z}{\left[(x-2)^2+(y-4)^2+(z+3)^2\right]^{3/2}}$$

$$= 180\times \frac{(x-2)\bar{a}_x+(y-4)\bar{a}_y+(z+3)\bar{a}_z}{\left[(x-2)^2+(y-4)^2+(z+3)^2\right]^{3/2}}$$

(b) Here point A(-3,2,0), putting the values in the above equation we get

$$E_A = 180\times \frac{(-3-2)\bar{a}_x+(2-4)\bar{a}_y+(0+3)\bar{a}_z}{\left[(-3-2)^2+(2-4)^2+(0+3)^2\right]^{3/2}}$$

$$= 180\times \frac{-5\bar{a}_x-2\bar{a}_y+3\bar{a}_z}{\left[5^2+2^2+3^2\right]^{3/2}}$$

$$= -3.84\,a_x - 1.54\,\bar{a}_y + 2.31\,\bar{a}_y \text{ V/m}$$

(c) $E_x = 1$ V/m is given, we get E_x setting this to 1.0 V/m.

$$\therefore\quad 1 = 180\times \frac{(x-2)}{\left[(x-2)^2+(y-4)^2+(z+3)^2\right]^{3/2}}$$

(or) $$180(x-2) = \left[(x-2)^2+(y-4)^2+(z+3)^2\right]^{3/2}$$

is the locus of all points on which $E_x = 1.0$ V/m

Similarly

along $E_y = 1$ V/m $$180(y-4) = \left[(x-2)^2+(y-4)^2+(z+3)^2\right]^{3/2}$$

along $E_z = 1$ V/m $$180(z+3) = \left[(x-2)^2+(y-4)^2+(z+3)^2\right]^{3/2}$$

A point charge Q_1 located at the origin in free space, produces a field $\bar{E} = 10\bar{a}_r$ V/m for $r = 1$m. (a) Find Q_1; (b) Using this value of Q_1, Find $\bar{E}$ at $P(2,-1,5)$ in Cartesian components. (c) Using the same value Q₁, find $\bar{E}$ at $C(2,2,2)$ in cylindrical ~~coordinates~~ components

Here $\bar{E} = 10\bar{a}_r$ V/m for $r = 1$m. The point charge is at the origin.

(a) $$\bar{E}(r) = \frac{Q_1}{4\pi\epsilon_0 r^2}\bar{a}_r$$

$$\therefore Q_1 = 4\pi\epsilon_0 r^2 |E(r)| = 4\pi \times 8.85 \times 10^{-12} \times (1)^2 \times 10$$
$$= 1.11156 \times 10^{-9} C$$
$$Q_1 = 1.11156 nC$$

(b) At point $P(2,-1,5)$, the vector originating from origin towards the point P and its magnitude are

$$\bar{r}_{op} = 2\bar{a}_x - \bar{a}_y + 5\bar{a}_z \quad \text{and} \quad |\bar{r}_{op}| = \sqrt{30}$$

The EFI at point P

$$\bar{E}_p = \frac{Q_1}{4\pi\epsilon_0 |\bar{r}_{op}|^2}\frac{\bar{r}_{op}}{|\bar{r}_{op}|} = \frac{1.11156 \times 10^{-9+12}}{4\pi \times 8.85}\ \frac{2\bar{a}_x - \bar{a}_y + 5\bar{a}_z}{(30)^{3/2}}$$

$$= 0.12\bar{a}_x - 0.06\bar{a}_y + 0.3045\bar{a}_z$$

(c) At point $C(2,2,2)$ the vector originating from origin towards the point C and the magnitude is

$$\bar{r}_{oc} = 2\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z \quad \text{and} \quad |\bar{r}_{oc}| = 2\sqrt{3}$$

The EFI at point C in Cartesian Coordinate System is

$$E_c = \frac{Q_1}{4\pi\epsilon_0 |\bar{r}_{oc}|^2}\frac{\bar{r}_{oc}}{|\bar{r}_{oc}|} = \frac{1.11156 \times 10^{-9+12}}{4\pi \times 8.85} \cdot \frac{2\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z}{(2\sqrt{3})^{3/2}}$$
$$= 0.48\bar{a}_x + 0.48\bar{a}_y + 0.48\bar{a}_z$$

converting $C(2,2,2)$ in cylindrical coordinate system (ρ, ϕ, z) yields

$$\rho = \sqrt{x^2+y^2} = \sqrt{2^2+2^2} = 2\sqrt{2}; \quad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{2} = 45$$
$$z = z$$

$$A_\rho = \cos\phi\, A_x + \sin\phi\, A_y$$
$$A_\phi = -\sin\phi\, A_x + \cos\phi\, A_y$$
$$A_z = A_z$$

$$E_{c\rho} = 0.48\cos\phi + 0.48\sin\phi$$
$$= 0.68$$

$$E_{c\phi} = -0.48\sin\phi + 0.48\cos\phi$$
$$= 0$$

$$E_{cz} = 0.48$$

$$\therefore E_{cy} = 0.68\,\bar{a}_\rho + 0.48\,\bar{a}_z \quad V/m.$$

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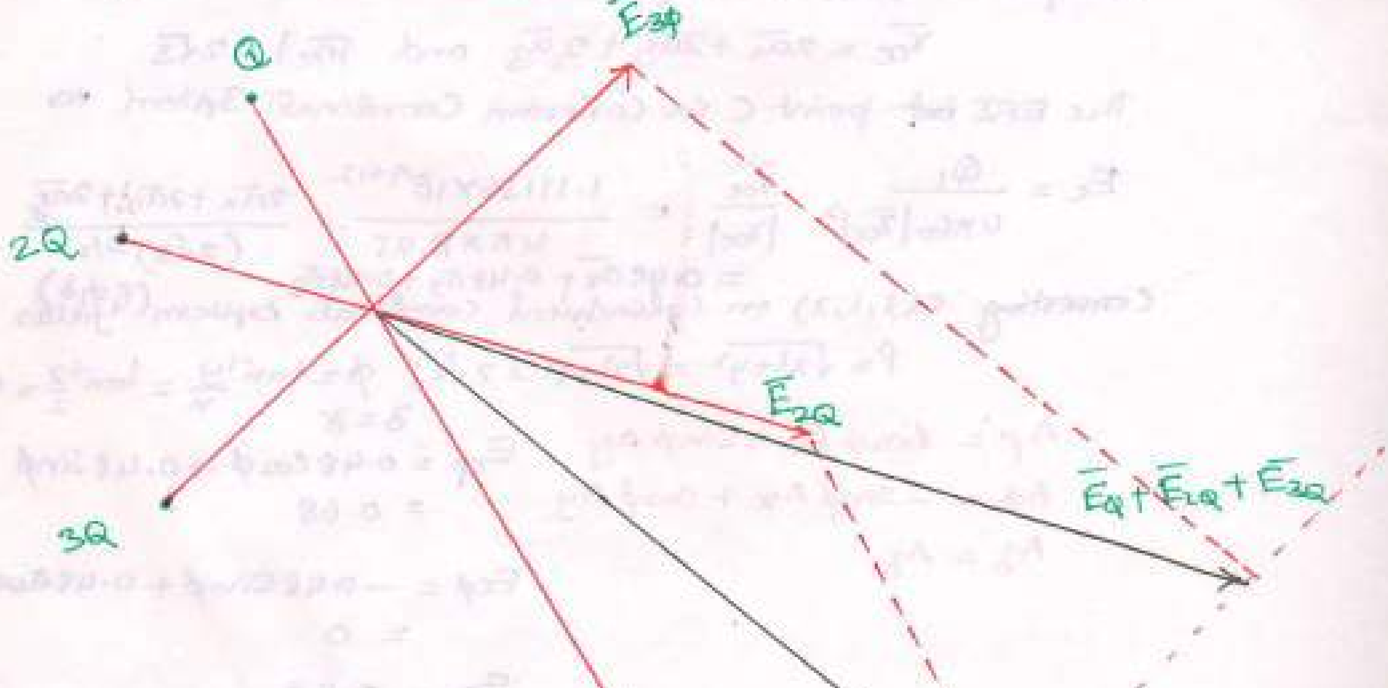
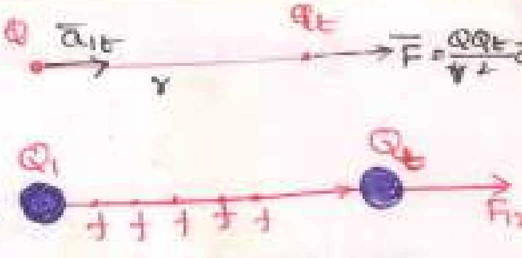
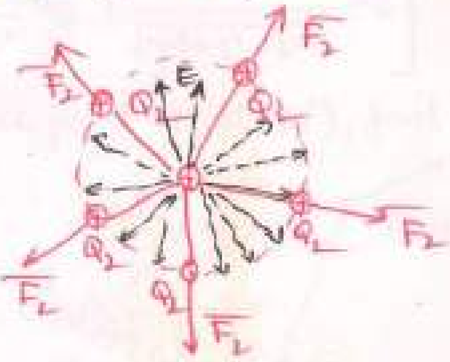
Super position Theorem.



Electric Field Intensity



If we now consider one charge fixed in position, say Q, and move a second charge slowly around, we note that there exists everywhere a force on this second charge; in other words, this second charge is displaying the existence of a force field. Call this second charge a test charge Q_t. The force on it is given by Coulomb's Law



$$\overline{F}_t = \frac{Q_1 Q_t}{4\pi\epsilon_0 R_{1t}^2}\overline{a}_{1t}$$

Writing this force per unit charge gives

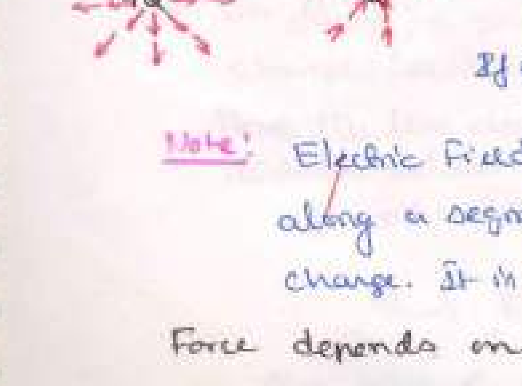
$$\frac{\overline{F}_t}{Q_t} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2}\overline{a}_{1t}$$

The quantity on right side of the equation is a function only of Q_1 and the directed line segment from Q_1 to the position of the test charge. This describes a vector field and is called the electric field intensity $\boxed{\overline{E}}$

Units volts/meter $\left(\frac{N}{C}\right)$

$$\therefore \quad E = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2}\overline{a}_{1t}$$

Def! The electric field intensity (or electric field strength) E is the force per unit charge when placed in an electric field.



$$\overline{E} = \frac{Q(\bar{r}-\bar{r}')}{4\pi\epsilon_0|\bar{r}-\bar{r}'|^3}$$

$$\overline{E} = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{N}\frac{Q_k(\bar{r}-\bar{r}_k)}{|\bar{r}-\bar{r}_k|^3}$$

$$\overline{F} = Q_t\overline{E}$$

If Q_t is large Force will be large

Note! Electric Field Intensity (E) is a vector quantity and is directed along a segment from the charge Q, to the position of any other charge. It is denoted by $\overline{E}$

Force depends on Q_t where ~~as~~ E does not

+ve charge will experience force in the direction of Electric field

1. A charge Q is situated at origin. Obtain the EFI at point (x, y, z).

Sol:- EFI $\overline{E} = \frac{Q}{4\pi\epsilon_0 R^2}\overline{a}_R$ V/m

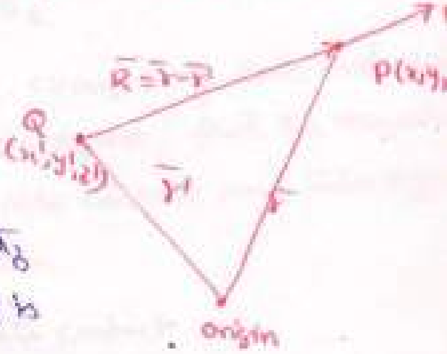
Here $\overline{R} = x\overline{a}_x + y\overline{a}_y + z\overline{a}_z$ and $|R| = \sqrt{x^2+y^2+z^2}$

and unit vector from origin to point (x, y, z) is $\overline{a}_R = \frac{x\overline{a}_x + y\overline{a}_y + z\overline{a}_z}{\sqrt{x^2+y^2+z^2}}$

Electric Field Intensity

$$\overline{E} = \frac{Q}{4\pi\epsilon_0(x^2+y^2+z^2)}\left\{\frac{x}{\sqrt{x^2+y^2+z^2}}\overline{a}_x + \frac{y}{\sqrt{x^2+y^2+z^2}}\overline{a}_y + \frac{z}{\sqrt{x^2+y^2+z^2}}\overline{a}_z\right]$$

2. In figure the charge is located at the source point Q(x', y', z'), find the field at a general point P(x, y, z)



Sol:- Here $\overline{r}' = x'\overline{a}_x + y'\overline{a}_y + z'\overline{a}_z$

$\overline{r} = x\overline{a}_x + y\overline{a}_y + z\overline{a}_z$

Thus $\overline{R} = \overline{r} - \overline{r}' = (x-x')\overline{a}_x + (y-y')\overline{a}_y + (z-z')\overline{a}_z$

The EFI at point P due to charge at point Q is

$$\overline{E}(r) = \frac{Q}{4\pi\epsilon_0}\frac{\overline{r}-\overline{r}'}{|\overline{r}-\overline{r}'|^3}$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{\left[(x-x')\overline{a}_x + (y-y')\overline{a}_y + (z-z')\overline{a}_z\right]}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}$$

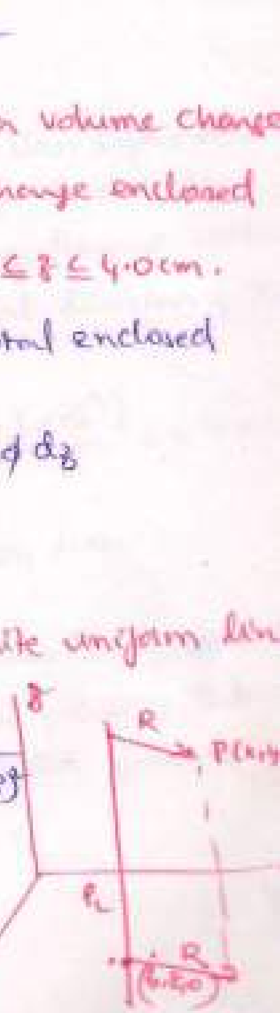
3. An electron beam of 2cm length is represented by a volume charge density, $\rho_v = -5\times10^{-6}\times e^{-10^5\rho z}$ C/m³. Determine charge enclosed in the region $0 \le \rho \le 1.0$ cm, $0 \le \phi \le 2\pi$ and $2.0 \le z \le 4.0$cm.

Sol: The integration over the cylinder yields the total enclosed charge.

$$Q = -\int_{0.02}^{0.04}\int_0^{2\pi}\int_0^{0.01} 5\times10^{-6}\times e^{-10^5\rho z}\,\rho\,d\rho\,d\phi\,dz$$

$$= -\frac{\pi}{40}\text{ PC}$$

4. In figure, a point P(x, y, z) is identified near an infinite uniform line charge located at x = 6, y = 8.



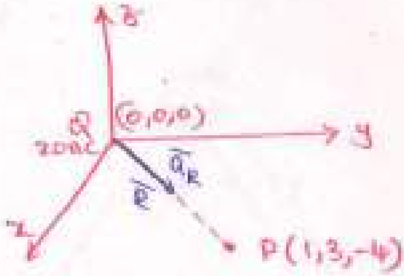
Here the line charge is at point (6, 8, 0). The radial distance between line charge and point P is $|\overline{R}| = \sqrt{(x-6)^2+(y-8)^2}$

Let unit vector $\overline{a}_\rho = \overline{a}_R = \frac{\overline{R}}{|R|} = \frac{(x-6)\overline{a}_x + (y-8)\overline{a}_y}{\sqrt{(x-6)^2+(y-8)^2}}$

Thus $\overline{E} = \frac{\rho_L}{2\pi\epsilon_0}\frac{(x-6)\overline{a}_x + (y-8)\overline{a}_y}{(x-6)^2+(y-8)^2}$

The field is not a function of z.

Eg: A point charge of 20nC is located at the origin. Determine the magnitude and direction of $\bar{E}$ at the point (1,3,-4)m



Sol:
$$E = \frac{Q}{4\pi\epsilon_0 |\bar{r}-\bar{r}_p|^3}(\bar{r}-\bar{r}_p)$$

$$= \frac{20\times10^{-9}}{4\pi\epsilon_0 (1^2+3^2+4^2)^{3/2}}(\bar{a}_x+3\bar{a}_y-4\bar{a}_z)$$

$$= \frac{20\times10^{-9}(\bar{a}_x+3\bar{a}_y+4\bar{a}_z)}{4\pi\epsilon_0(\sqrt{26})^3}$$

$$= 1.3558\bar{a}_x + 4.0676\bar{a}_y + 5.4235\bar{a}_z \text{ V/m.}$$

$\therefore |E| = 6.9136$ V/m. and its direction is $\bar{a}_R$.

Eg: Two small identical conducting spheres have charges of 2nc and -1nc respectively. when they are separated by 4cm apart, find the magnitude of the force between them. If they are brought into contact and then again separated by 4cm find the force between them.

Sol:- Case 1 Before the charges are brought into contact

$$|F| = \left|\frac{Q Q_2}{4\pi\epsilon_0 R_{12}^2}\right| \quad \text{where } R_{12} = 4\text{cm} = 4\times10^{-2}\text{ m}$$

$$= \left|\frac{2\times10^{-9}\times(-1\times10^{-9})}{4\pi\epsilon_0\times(4\times10^{-2})^2}\right| = 11.234\,\mu N$$

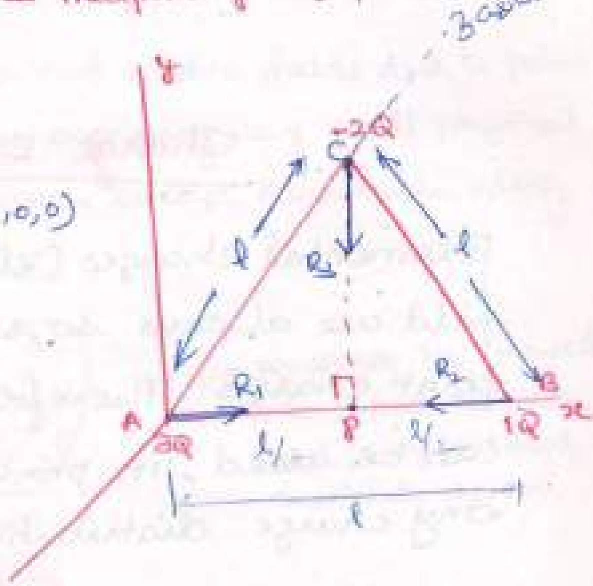
Case 2: The charges are brought into contact and then separated when charges are brought into contact, the charge distribution takes place due to the transfer of charge. The transfer of charge continues till both charges attain same value due to equal division of the two charges.

$$\text{Charge on each sphere} = \frac{Q_1+Q_2}{2} = \frac{2\times10^{-9}+(-1\times10^{-9})}{2} = 0.5nC$$

$$\therefore |F| = \left|\frac{Q Q_2}{4\pi\epsilon_0 R_{12}^2}\right| = \left|\frac{0.5\times10^{-9}\times0.5\times10^{-9}}{4\pi\epsilon_0\times(4\times10^{-2})^2}\right| = 1.404\,\mu N.$$

Note: Initially before charges are brought together the force between them was attractive as charges are of opposite polarity. But when they brought in contact and then separated, the force is repulsive in nature.

Example: Point charges are located at each corner of an equilateral triangle. If the charges are 3Q, −2Q, and 1Q, find E at midpoint of 3Q & 1Q side.



Solution: Let $AB = BC = CA = l$

$CP = \frac{\sqrt{3}l}{2}$

$A(0,0,0)$, $B(l,0,0)$, $C\left(\frac{l}{2}, \frac{\sqrt{3}l}{2}, 0\right)$ and $P\left(\frac{l}{2}, 0, 0\right)$

$\bar{E}$ at P is to be obtained.

$$\bar{E}_1 = \frac{Q_1}{4\pi\epsilon_0 R_1} \bar{a}_{R_1}$$

$$\bar{R}_1 = \left(\frac{l}{2} - 0\right)\bar{a}_x + 0\bar{a}_y + 0\bar{a}_z$$

$$= 0.5l\bar{a}_x,$$

$$|\bar{R}_1| = 0.5l$$

$$\therefore \bar{a}_{R_1} = \frac{\bar{R}_1}{|\bar{R}_1|} = \frac{0.5l\bar{a}_x}{0.5l} = \bar{a}_x$$

$$\therefore \bar{E}_1 = \frac{3Q}{4\pi\epsilon_0 (0.5l)^2}\bar{a}_x = \frac{1.078\times10^{11} Q}{l^2}\bar{a}_x$$

Now $\bar{E}_2 = \frac{Q_2}{4\pi\epsilon_0 R_2}\bar{a}_{R_2}$,

$$\bar{R}_2 = \left(\frac{l}{2} - l\right)\bar{a}_x + 0\bar{a}_y + 0\bar{a}_z = -0.5l\bar{a}_x$$

$$|\bar{R}_2| = 0.5l$$

$$\therefore \bar{a}_{R_2} = \frac{\bar{R}_2}{|\bar{R}_2|} = -\bar{a}_x$$

$$\bar{E}_2 = \frac{1Q}{4\pi\epsilon_0 (0.5l)^2}(-\bar{a}_x) = \frac{-3.595\times10^{10} Q}{l^2}\bar{a}_x$$

and $\bar{E}_3 = \frac{Q_3}{4\pi\epsilon_0 R_3}\bar{a}_{R_3}$, $\bar{R}_3 = \left(\frac{l}{2} - \frac{l}{2}\right)\bar{a}_x + \left(0 - \frac{\sqrt{3}}{2}l\right)\bar{a}_y + 0\bar{a}_z$

$$\bar{R}_3 = 0.866l\bar{a}_y, \quad |\bar{R}_3| = 0.566l,$$

$$\bar{a}_{R_3} = \frac{\bar{R}_3}{|\bar{R}_3|} = -\bar{a}_y$$

$$\bar{E}_3 = \frac{-2Q}{4\pi\epsilon_0 (0.866l)^2}(-\bar{a}_y) = \frac{2.3968\times10^{10}}{l^2}Q\cdot\bar{a}_y$$

$\bar{E}$ at P $= \bar{E}_1 + \bar{E}_2 + \bar{E}_3$

$$= \frac{Q}{l^2}\left[7.185\times10^{10}\bar{a}_x + 2.3968\times10^{10}\bar{a}_y\right] \text{ V/m}.$$

Charge Distribution

Elemental charges (electrons and protons) that create electric field are always so small that they can be considered as point charges. Therefore the equation $\overline{E} = \sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0} \frac{Q_i}{R_i^2} \overline{a}_i$ V/m can be used, in principle, to calculate the electric field of any charge distribution.

However, even for a tiny amount of charge, the number of elemental charges is very large. For example:

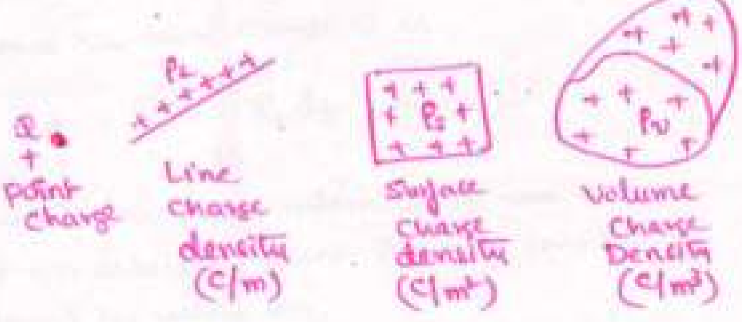
-1 pC (-10^{-12}C) contains about 10^7 electrons.

Charge of electron -1.602×10^{-19}C $\quad \therefore \frac{-1\text{pC}}{-1.602 \times 10^{-19}\text{C}} = 10^7$ electrons

Therefore, in macroscopic electromagnetism, it is convenient to introduce the concept of charge density, and then to use integral calculus to evaluate the field of a charge distribution.

Electric Fields Due to CONTINUOUS CHARGE DISTRIBUTIONS

Sofar we have considered only forces and electric fields due to point charges, which are essentially charges occupying very small physical space. It is possible to have continuous charge distribution along a line, on a surface, or in a volume.



ρ — must not be confused with radial distance used in cylindrical coordinate system.

The charge element dQ and the total charge Q due to these charge distributions are

$Q = \rho L$ L is the length of the line.

$$dQ = \rho_L dl$$

$$Q = \int_L \rho_L dL \rightarrow \text{line charge}$$

$$dQ = \rho_S ds$$

$$Q = \int_S \rho_S ds \longrightarrow \text{Surface charge}$$

$$dQ = \rho_v dv$$

$$Q = \int_v \rho_v dv \longrightarrow \text{Volume charge.}$$

Electric field intensity due to each of the charge distribution ρ_L, ρ_S, & ρ_v may be regarded as the summation of the field contributed by the numerous point charges making up the charge distribution.

Differential Electric Field.

$$d\overline{E} = \frac{\rho_L dL}{4\pi\epsilon_0 R^2}\overline{a_R} \qquad \therefore \quad \overline{E} = \int_L \frac{\rho_L dl}{4\pi\epsilon_0 R^2}\overline{a}_R \quad \text{(Line charge)}$$

$$d\overline{E} = \frac{\rho_S ds}{4\pi\epsilon_0 R^2}\overline{a_R} \qquad \overline{E} = \int_S \frac{\rho_S ds}{4\pi\epsilon_0 R^2}\overline{a}_R \quad \text{(Surface charge)}$$

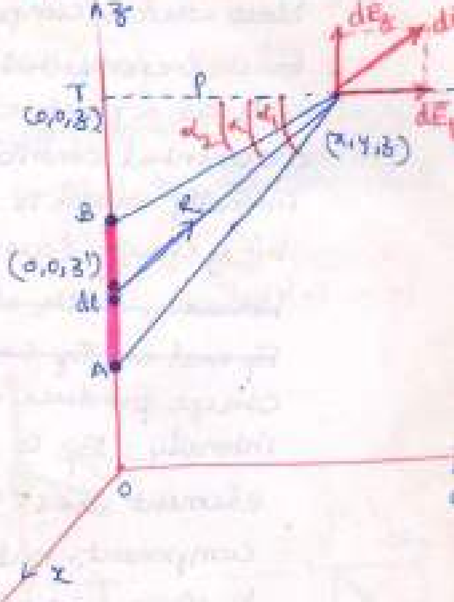
$$d\overline{E} = \frac{\rho_v dv}{4\pi\epsilon_0 R^2}\overline{a_R} \qquad \overline{E} = \int_v \frac{\rho_v dv}{4\pi\epsilon_0 R^2}\overline{a}_R \quad \text{(volume charge)}$$

It should be noted that R^2 and $\overline{a}_R$ vary as the integrals are evaluated.

We shall now apply these formulas to some specific charge distributions.

1. LINE CHARGE.

We desire the electric field intensity $\overline{E}$ at any and every point resulting from a uniform line charge density ρ_L



Consider a line charge with uniform charge density ρ_L extending from A to B along the z-axis as shown in figure. The charge element dQ associated with element $dl = dz$ of the line is

$$dQ = \rho_L dl = \rho_L dz$$

Hence the total charge Q is

$$Q = \int_{z_A}^{z_B} \rho_L dz. \longrightarrow ①$$

The electric field intensity $\overline{E}$ ~~can be~~ at an arbitrary point $P(x,y,z)$ can be found by using eq

$$\overline{E} = \int_L \frac{\rho_L dl}{4\pi\epsilon_0 R^2}\overline{a}_R \longrightarrow ②$$

It is customary to denote the field point at which the field is to be evaluated (x', y', z').

From the figure we choose that $dl = dz'$

$$\left.\begin{aligned} \overline{R} &= (x,y,z) - (0,0,z') = x\overline{a}_x + y\overline{a}_y + (z-z')\overline{a}_z \\ \text{or } \overline{R} &= \rho\overline{a}_\rho + (z-z')\overline{a}_z \\ R^2 &= |R|^2 = x^2+y^2+(z-z')^2 = \rho^2+(z-z')^2 \end{aligned}\right\} \longrightarrow ③$$

$$\frac{a_R}{R^2} = \frac{R^2}{|R|^3} = \frac{\rho\overline{a}_\rho + (z-z')\overline{a}_z}{\left[\rho^2+(z-z')^2\right]^{3/2}}$$

Substituting ③ into equation ②, we get

$$\overline{E} = \frac{\rho_L}{4\pi\epsilon}\int \frac{\rho\overline{a}_\rho + (z-z')\overline{a}_z}{\left[\rho^2+(z-z')^2\right]^{3/2}}dz' \longrightarrow ④$$

To evaluate this, it is convenient that we define $\alpha, \alpha_1, \alpha_2$ as in figure

$$\overline{R} = \left[\rho^2 + (z-z')^2\right]^{1/2} = \rho\sec\alpha$$

$$OT = z$$

$$z' = OT - \rho\tan\alpha$$

$$dz' = -\rho\sec^2\alpha\, d\alpha$$

$$\left(\because \left[\rho^2 + (z - z - \rho\tan\alpha)^2\right]^{1/2} = \left[\rho^2 + (\rho\tan\alpha)^2\right]^{1/2} = \left[\rho^2(1+\tan^2\alpha)\right]^{1/2} = (\rho^2\sec^2\alpha)^{1/2} = \rho\sec\alpha.\right)$$

Hence the equation ④ becomes

$\alpha_2 = -\pi/2$
$\alpha_1 = \pi/2$

$$\overline{E} = \frac{-\rho_L}{4\pi\epsilon_0}\int_{\alpha_1}^{\alpha_2} \frac{\rho\sec^2\alpha\left[\cos\alpha\,\overline{a}_\rho + \sin\alpha\,\overline{a}_z\right]d\alpha}{\rho^2\sec^2\alpha}$$

$$= \frac{-\rho_L}{4\pi\epsilon_0\rho}\int_{\alpha_1}^{\alpha_2}\left[\cos\alpha\,\overline{a}_\rho + \sin\alpha\,\overline{a}_z\right]d\alpha$$

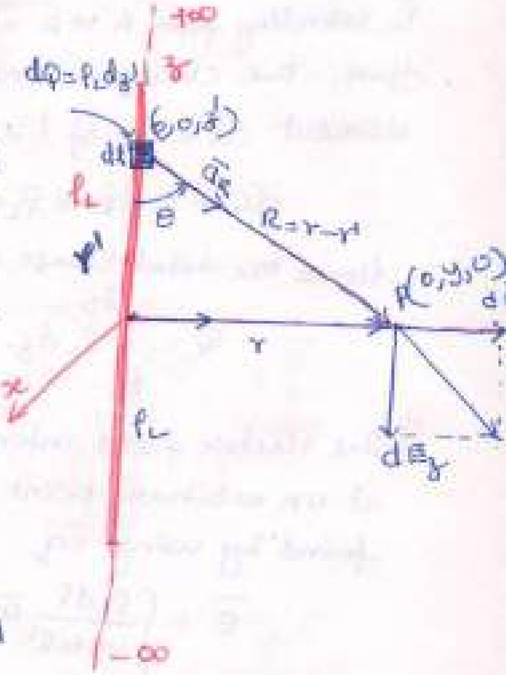
$$\overline{E} = \frac{-\rho_L}{4\pi\epsilon_0}\int_{\alpha_1}^{\alpha_2}\frac{\rho(a_\rho + \tan\alpha}{(\rho\sec\alpha)^3}$$

$$= \frac{-\rho_L}{4\pi\epsilon_0}\int_{\alpha_1}^{\alpha_2}\frac{\rho^2\sec^2\alpha\,(a_\rho + \tan}{\rho^3\sec^3\alpha}$$

$$= \frac{-\rho_L}{4\pi\epsilon_0\rho}\int_{\alpha_1}^{\alpha_2}\frac{\overline{a}_\rho + \tan\alpha\,\overline{a}_z}{\sec\alpha}$$

$$= \frac{-\rho_L}{4\pi\epsilon_0\rho}\int_{\alpha_1}^{\alpha_2}\overline{a}_\rho\cos\alpha\,\overline{a}_\rho +$$

In electric field Intensity
Now which component ~~of~~ are present?



Each incremental length of line charge acts as a point charge and produces an incremental contribution to the electric field intensity which is directed away from the bit of charge (assuming a positive line charge) ~~However, each element does produce an E_ρ and an E_z component,~~ No element of charge produce an ϕ component of electric intensity; E_ϕ is zero. However, each element does produce an E_ρ and an E_z component but the contribution to E_z by elements of charge which are equal distances above and below the point at which we are determining the field will cancel.

∴ we have to find that only an E_ρ component that varies with ρ. To find this, we choose a point $P(0,y,0)$ on the y-axis at which determine the field, this point lacks variation of field with ϕ & z

∴ The incremental field at P due to the incremental charge

$$dQ = \rho_L dz'$$

and

$$d\bar{E} = \frac{\rho_L dz' (\bar{r}-\bar{r}')}{4\pi\epsilon_0 |r-r'|^3}$$

where

$$\bar{r} = y\bar{a}_y = \rho\bar{a}_\rho$$

$$r' = z'\bar{a}_z$$

and

$$\bar{r}-\bar{r}' = \rho\bar{a}_\rho - z'\bar{a}_z$$

$$\therefore d\bar{E} = \frac{\rho_L dz'(\rho\bar{a}_\rho - z'\bar{a}_z)}{4\pi\epsilon_0(\rho^2+z'^2)^{3/2}}$$

Since only the $\bar{E}_\rho$ component is present, we may simplify

$$d\bar{E}_\rho = \frac{\rho_L \rho\, dz'}{4\pi\epsilon_0(\rho^2+z'^2)^{3/2}}$$

$-E_z$ & E_z [illegible] equal and cancel

Since ~~cos~~ $E_z = 0$

and $$\bar{E}_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho\, dz'}{4\pi\epsilon_0(\rho^2+z'^2)^{3/2}}$$

Integrating by integral tables or change of variables

$z' = \rho\cot\theta$ we have.

$$E_\rho = \frac{\rho_L}{4\pi\epsilon_0}\rho\left(\frac{1\, z'}{\rho^2\sqrt{\rho^2+z'^2}}\right)^{\infty}$$

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0\rho}$$

(or) $z' = \rho\cot\theta$, $dz' = -\rho\cos$

Since $R = \rho\csc\theta$, our integ

become

$$dE_\rho = \frac{\rho_L dz'}{4\pi\epsilon_0 R^2}\sin\theta = \frac{-\rho_L \sin\theta}{4\pi\epsilon}$$

$$E_\rho = \frac{-\rho_L}{4\pi\epsilon_0\rho}\int_{\pi}^{0}\sin\theta\, d\theta = \frac{\rho_L}{4\pi\epsilon}$$

$$= \frac{\rho_L}{2\pi\epsilon_0\rho}$$

(or) $$\bar{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\bar{a}_\rho$$

Thus for finite line charge

$$\bar{E} = \frac{\rho_L}{4\pi\epsilon_0 \rho}\left[-(\sin\alpha_2 - \sin\alpha_1)\bar{a}_\rho + \underbrace{(\cos\alpha_2 - \cos\alpha_1)\bar{a}_z}\right]$$

Substituting the values of $\alpha_1 = -\pi/2$, $\alpha_2 = \pi/2$ this term vanishes to zero.

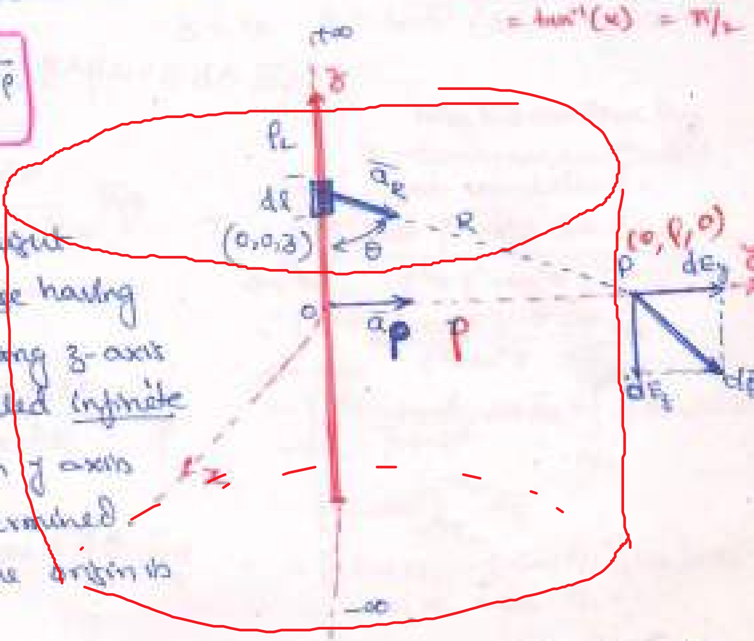
As a special case, for an infinite line charge, point B is at $(0,0,\infty)$ and A at $(0,0,-\infty)$ so that $\alpha_1 = \pi/2$ & $\alpha_2 = -\pi/2$; the z-component vanishes and above equation becomes

$z' = z$-plane
$\alpha = \tan^{-1}(-\infty) = -\pi/2$
$= \tan^{-1}(\infty) = \pi/2$

$$\boxed{\bar{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho}\bar{a}_\rho}$$

<u>Alternatively</u>

Consider an infinitely long straight line carrying uniform line charge having density ρ_L c/m. Let this lies along z-axis from $-\infty$ to $+\infty$ and hence called <u>infinite line charge</u>. Let point P is on y axis at which EFI is to be determined. The distance of point P from the origin is 'r' as shown in figure.



Consider a small differential length dl carrying a charge dQ along the line as shown. $\therefore$ $dl = dz$. ($\because$ the charge dQ is along z-axis)

$$\therefore dQ = \rho_L dl = \rho_L dz$$

The coordinates of dQ are (0,0,z) while the coordinates of point P are (0,ρ,0). Hence the distance vector $\bar{R}$ can be written as

$$\bar{R} = \bar{r}_p - \bar{r}_{dl} = [\rho\bar{a}_\rho - z\bar{a}_z]$$

$$\therefore |\bar{R}| = \sqrt{\rho^2+z^2} \qquad \therefore \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{\rho\bar{a}_\rho - z\bar{a}_z}{\sqrt{r^2+z^2}}$$

$$\therefore d\bar{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\bar{a}_R$$

$$= \frac{\rho_L dz}{4\pi\epsilon_0(\sqrt{\rho^2+z^2})^2}\left[\frac{\rho\bar{a}_\rho - z\bar{a}_z}{\sqrt{\rho^2+z^2}}\right] \longrightarrow$$

mathematically z-component will be cancelled.

Note: For every charge on positive z axis, there is equal charge present on negative z-axis. Hence the z-component of electric fie intensities produced by such charges at point P will cancel eac other. Hence effectively there will not be any z component of $\bar{E}$ a P.

Hence the equation of $d\bar{E}$ can be written by eliminating $\bar{a}_z$ compone

$$\therefore d\bar{E} = \frac{\rho_L dz}{4\pi\epsilon_0(\sqrt{\rho^2+z^2})^2}\frac{\rho\bar{a}_y}{\sqrt{\rho^2+z^2}}$$

Now by integrating $d\bar{E}$ over the z-axis from $-\infty$ to ∞ we can obtain total $\bar{E}$ at point P.

$$\therefore \bar{E} = \int_{-\infty}^{\infty} \frac{\rho_L}{4\pi\epsilon_0(\rho^2+z^2)^{3/2}}\rho\, dz\, \bar{a}_y$$

$z = \rho\tan\theta$

$dz = \rho\sec^2\theta\, d\theta.$

for $z=-\infty$ $\theta = \tan^{-1}(-\infty) = -\pi/2 = -90°$

$z=\infty$ $\theta = \tan^{-1}(\infty) = \pi/2 = 90°$

$$\therefore \bar{E} = \int_{-\pi/2}^{\pi/2} \frac{\rho_L}{4\pi\epsilon_0(\rho^2+\rho^2\tan^2\theta)^{3/2}}\rho\,\rho\sec^2\theta\, d\theta\, \bar{a}_y$$

$$= \frac{\rho_L}{4\pi\epsilon_0}\int_{-\pi/2}^{\pi/2} \frac{\rho^2\sec^2\theta\, d\theta}{\rho^3\,[1+\tan\theta]^{3/2}}\bar{a}_\rho \quad + \quad \int_{\theta=-\pi/2}^{\pi/2}\frac{\rho\tan\theta\,\rho\sec^2\theta\, d\theta\, \bar{a}_z}{(\rho^2+\rho^2\tan^2\theta)^{3/2}}$$

Mathematically z-component will get canceled.

$$= \int_{\theta=-\pi/2}^{\pi/2}\frac{\rho\tan\theta\,\rho\sec^2\theta\, d\theta}{\rho^3\sec^3\theta}\bar{a}_z$$

$$= \int_{-\pi/2}^{\pi/2}\frac{\tan\theta}{\rho\sec\theta}d\theta\,\bar{a}_z = \int_{-\pi/2}^{\pi/2}\frac{1}{\rho}\sin\theta\, d\theta\,\bar{a}_z$$

$$= [-\cos\theta]_{-\pi/2}^{\pi/2}\bar{a}_z$$

$$= [-\cos\pi/2 - (-\cos-\pi/2)]\bar{a}_z = 0.$$

$$= \cos\pi/2 = \cos-\pi/2.$$

But $1+\tan^2\theta = \sec^2\theta$

$$\bar{E} = \frac{\rho_L}{4\pi\epsilon_0}\int_{-\pi/2}^{\pi/2}\frac{\sec^2\theta\, d\theta}{\rho\sec^3\theta}\bar{a}_\rho$$

$$= \frac{\rho_L}{4\pi\epsilon_0}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta\, \bar{a}_\rho$$

$$= \frac{\rho_L}{4\pi\epsilon_0\rho}[\sin\theta]_{-\pi/2}^{\pi/2}\bar{a}_\rho = \frac{\rho_L}{4\pi\epsilon_0\rho}\left[\sin\frac{\pi}{2} - \sin\frac{-\pi}{2}\right]\bar{a}_\rho$$

$$= \frac{\rho_L}{4\pi\epsilon_0\rho}[1-(-1)]\bar{a}_\rho = \frac{2\rho_L}{4\pi\epsilon_0\rho}\bar{a}_\rho$$

$$\boxed{\therefore \bar{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\bar{a}_\rho \ \ V/m}$$

Note: 1) The field intensity $\bar{E}$ at any point has no component in the direction parallel to the line along which the charge is located and the charge is infinite. For example if line charge is parallel to z-axis, $\bar{E}$ can not have $\bar{a}_z$ component, if line charge is parallel to y-axis, $\bar{E}$ can not have $\bar{a}_\rho$ component. This makes the integration calculations easy.

2) The above equation consists of ρ and $\bar{a}_\rho$ which do not have any meaning in cylindrical coordinate system. The distance ρ is to be obtained by distance formula while $\bar{a}_\rho$ is unit vector in the direction of $\bar{\rho}$.

A point charge of 5nC is located at (-3,4,0), while ~~y=~~ line y=1 z=1 carries uniform charge 2nC/m

(a) If V=0 V at O(0,0,0) find V at A(5,0,1)

(b) If V=100V at B(1,7,1) find V at C(-2,5,3)

Let the potential at a point P be

$$V_L = -\int E\,dl = -\int \frac{\rho_L}{2\pi\varepsilon_0 \rho}\,\bar{a}_\rho \cdot d\rho\,\bar{a}_\rho = -\frac{\rho_L}{2\pi\varepsilon_0}\ln\rho + C_2$$

$$V = V_Q + V_L$$

$$= -\frac{\rho_L}{2\pi\varepsilon_0}\ln\rho + \frac{Q}{4\pi\varepsilon_0 r} + C_1 + C_2$$

(a) If V=0 at O(0,0,0) and V at A(5,0,1) is to be determined

$$\rho = |(x,y,z) - (x,1,1)| = \sqrt{(y-1)^2 + (z+1)^2}$$

$$\rho_O = |(0,0,0) - (0,1,1)| = \sqrt{2}$$

$$\rho_A = |(5,0,1) - (5,1,1)| = 1$$

$$r_O = |(0,0,0) - (-3,4,0)| = 5$$

$$r_A = |(5,0,1) - (-3,4,0)| = 9$$

Hence $V_O - V_A = \frac{-\rho_L}{2\pi\varepsilon_0}\ln\frac{\rho_O}{\rho_A} + \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_O} - \frac{1}{r_A}\right]$

$$0 - V_A = -36\ln\sqrt{2} + 45\left(\frac{1}{5} - \frac{1}{9}\right)$$

$$V_A = 36\ln\sqrt{2} - 4 = 8.477\text{V}$$

(b) If V=100V at B(7,2,1) and V at C(-2,5,3) is to be determined

$\rho_B = 1$

$\rho_C = \sqrt{20}$

$r_B = \sqrt{21}$

$r_C = \sqrt{11}$

$$V_C - V_B = \frac{-\rho_L}{2\pi\varepsilon_0}\ln\frac{\rho_o}{\rho_B} + \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_C} - \frac{1}{r_B}\right)$$

$$V_C - 100 = -36\ln\frac{\sqrt{20}}{1} + 45\left[\frac{1}{\sqrt{11}} - \frac{1}{\sqrt{21}}\right]$$

$V_C - 100$ ~~=50.~~ -50.175

or $V_C = 49.825$

$$\therefore V_{BC} = V_C - V_B = 49.825 - 100$$

$$= -50.175$$

A uniform line charge density of 20nC/m lies on the z-axis between z=1 and z=3m. No other charge is present. Find E at (a) the origin (b) P(4,0,0)

Sol:- Line charge $\rho_L = 20nC/m$ on z-axis $z=1$ & $z=3$

(a) EFI at origin (0,0,0)

$$\bar{E}_z = -\int_1^3 \frac{\rho_L \, dz'}{4\pi\epsilon_0 z'^2} = \frac{20\times1000}{4\pi\times8.55} \times \frac{1}{z'}\Big|_1^3 = \frac{179.92}{z'}\Big|_1^3$$

$$= +119.95 V/m.$$

Thus $E = +119.95\bar{a}_z = (0, 0, +119.95)$ V/m.

(b) ~~Electric~~ EFI at P(4,0,0)

the radial distance between the line charge $\rho_L = 20nC/m$ on z-axis $z=1$ and $z=3$, and point P(4,0,0) is

$$\bar{R} = \sqrt{(4-0)^2+(0-z)^2} = \sqrt{4^2+z^2}$$

$$\bar{a}_R = \frac{\bar{R}}{|R|} = \frac{(4-0)\bar{a}_y + (0-z)\bar{a}_z}{\sqrt{4^2+z^2}} = \frac{4\bar{a}_y - z\bar{a}_z}{\sqrt{4^2+z^2}}$$

Thus
$$\bar{E}_L = \int_1^3 \frac{20\times10^{-9}\times9\times10^{+9}\,(4\bar{a}_y - z'\bar{a}_z)\,dz'}{(16+z'^2)^{3/2}}$$

$$= \int_1^3 \frac{719.71\bar{a}_y - 179.72\,z'\bar{a}_z}{(16+z'^2)^{3/2}}\,dz'$$

$$\frac{\rho_L \int^{z'} dz'}{4\pi\epsilon_0 z_1}.$$

$$\frac{20\times10^{-9}}{4\pi\epsilon_0} \; 2.$$

$dz' = 4\,\mathrm{cosec}^2\theta\, d\theta.$

$$\int_1^3 \frac{719.71\,\bar{a}_z\,(4\,\mathrm{cosec}^2\theta)\,d\theta}{16(1+\cot^2\theta)^{3/2}} = \int_1^3 \frac{719.71\,\bar{a}_z\,(-4\,\mathrm{cosec}^2\theta\, d\theta)}{16(\mathrm{cosec}^3\theta)} = \int_1^3 \frac{719.71}{4\,\mathrm{cosec}\,\theta}$$

Integrating separately

$$\bar{E}_y = \int_1^3 \frac{719.71\bar{a}_y}{(16+z'^2)^{3/2}}\,dz'$$

putting $z' = 4\cot\theta$ & integrating by change of variables.

$$E_y = 719.71\,\frac{z'}{16\sqrt{16+3^2}}\Big|_1^3 = 719.71\times\frac{1}{16}\left[\frac{3}{5} - \frac{1}{\sqrt{17}}\right]$$

$$= 16.08$$

$dz' = -4\,\mathrm{cosec}^2\theta\, d\theta.$

$\int \frac{\sin\theta\, d\theta}{4}.$

$d\theta = \frac{dz\,\sin^2\theta}{4}.$

$\int \frac{\sin\theta \sin^2\theta\, ~~dz~~}{4}.$

$$E_z = -179.72\int_1^3 \frac{z'}{(16+z'^2)^{3/2}}\,dz'$$

$$= -7.64.$$

∴ EFI at P(4,0,0) is $\bar{E}_P = 16.08\,\bar{a}_y - 7.64\,\bar{a}_z$ V/m.

A line charge density of 24 nc/m is located in free space on the line $y=1, z=2$. (a) Find $\bar{E}$ at $P(6,-1,3)$ (b) what point charge Q_A should be located at $A(-3,4,1)$ to cause E_y to be equal to zero at P?

Sol: The line charge density $\rho_L = 24$ nc/m at $y=1$, & $z=2$.

(a) Electric Field Intensity at $P(6,-1,3)$ is calculated as below.

The radial distance between the line charge & point $P(6,-1,3)$

$$R = \sqrt{(-1-1)^2+(3-2)^2} \quad (\because \text{ it is } yz\text{-plane})$$

$$= \sqrt{2^2+1^2} = \sqrt{5}$$

$$\therefore \bar{E}_L = \frac{\rho_L}{2\pi\epsilon_0 R}\bar{a}_R \quad \text{where } \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{(-1-1)a_y + (3-2)\bar{a}_z}{\sqrt{3}}$$

$$= \frac{-2\bar{a}_y + \bar{a}_z}{\sqrt{5}}$$

$$\therefore E_L = \frac{24\times10^{-9}\times9\times10^{9}\,(-2\bar{a}_y+\bar{a}_z)}{\sqrt{5}} = -172.76\bar{a}_y + 86.37\bar{a}_z \text{ V/m}$$

(b) Let Q_A be located at $A(-3,4,1)$ for E_y at point P to be zero

$$\therefore r_{AP} = \bar{r}_P - \bar{r}_A = (6+3)\bar{a}_x + (-1-4)\bar{a}_y + (3-1)\bar{a}_z$$

$$= 9\bar{a}_x - 5\bar{a}_y + 2\bar{a}_z$$

Its magnitude is

$$|\bar{r}_{AP}| = \sqrt{9^2+5^2+2^2} = \sqrt{81+25+4} = \sqrt{110}$$

$$\bar{E}_Q = \frac{Q_A}{4\pi\epsilon_0}\frac{\bar{r}_{AP}}{|\bar{r}_{AP}|^{3/2}} = \frac{Q_A}{4\pi\epsilon_0}\frac{9\bar{a}_x - 5\bar{a}_y + 2\bar{a}_z}{(110)^{3/2}}$$

$$\frac{Q_A}{4\pi\epsilon_0}\frac{(-5\bar{a}_y)}{110^{3/2}} = 172.76\,\bar{a}_y$$

$$Q_A = \frac{-4\pi\epsilon_0\times172.76\times(110)^{3/2}}{5}$$

$$Q_A = \underline{-4.43\,\mu C}$$

2. A SURFACE CHARGE

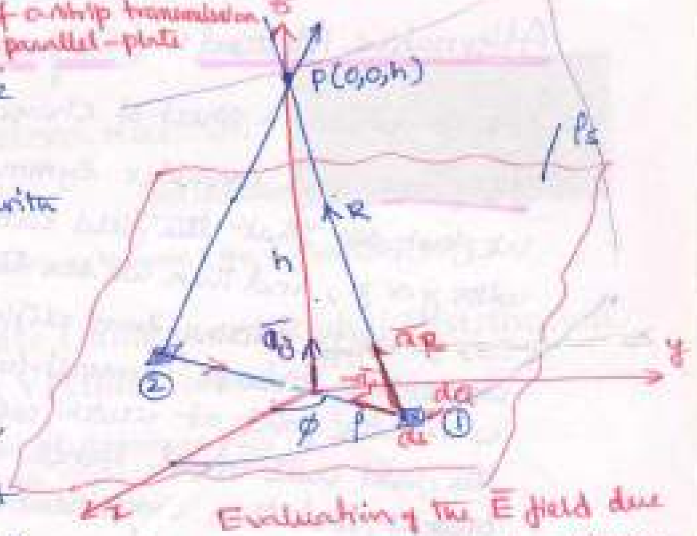
This configuration often used to approximate charge on conductors of a strip transmission line or a parallel-plate capacitor.

Consider an infinite sheet of charge in the xy-plane with uniform charge density ρ_s. The charge associated with an elemental area ds is

Static charge resides on conductor surface and not in their interiors; for this reason ρ_s commonly known as surface charge density

$$dQ = \rho_s\, ds$$

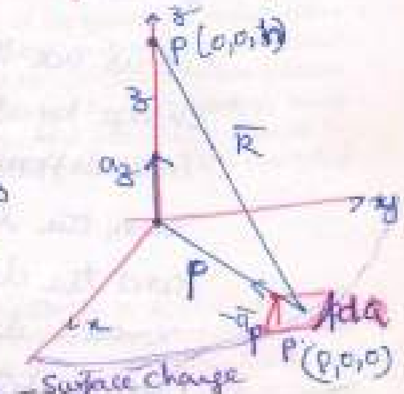
from equation $\bar{E} = \int_S \frac{\rho_s\, ds}{4\pi\epsilon_0 R^2}\bar{a}_R$,



Evaluating the $\bar{E}$ field due to an infinite sheet of charge.

the contribution to the $\bar{E}$ field at point P (0, 0, h) by the charge dQ on the elemental surface 1 shown in figure

$$d\bar{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\bar{a}_R \longrightarrow ①$$



from the figure

$$(0-\rho)\bar{a}_\rho + (h-0)\bar{a}_z = \rho(-\bar{a}_\rho) + h\bar{a}_z$$

$$\bar{R} = \rho(-\bar{a}_\rho) + h\bar{a}_z$$

$$R = |\bar{R}| = (\rho^2+h^2)^{1/2}$$

$$\bar{a}_R = \frac{\bar{R}}{R}, \quad dQ = \rho_s ds = \rho_s \rho\, d\phi\, d\rho$$

ρ_s – surface charge

$ds = \rho d\phi d\rho$

Substituting these terms into eq ① gives

$$d\bar{E} = \frac{\rho_s \rho\, d\phi\, d\rho\left[-\rho\bar{a}_\rho + h\bar{a}_z\right]}{4\pi\epsilon_0\left[\rho^2+h^2\right]^{3/2}} = \frac{\rho_s \rho\, d\phi\, d\rho\;\; h\bar{a}_z}{4\pi\epsilon_0(\rho^2+h^2)^{3/2}}$$

$$\bar{a}_\rho = \int_0^{2\pi} \cos\phi\,\bar{a}_x + \sin\phi\,\bar{a}_y = 0$$

Owing to symmetry of the charge distribution for every element 1, there is a corresponding element 2 whose contribution along $\bar{a}_\rho$ cancels that of element 1, as illustrated in fig. Thus contribution to E_ρ add up to zero so that $\bar{E}$ has only z-component. This also can be shown mathematically by replacing $\bar{a}_\rho$ with $\cos\phi\,\bar{a}_x + \sin\phi\,\bar{a}_y$. Integration of $\cos\phi$ or $\sin\phi$ over $0<\phi<2\pi$ gives zero. Therefore

$E_\rho = 0$

a_ρ cancels

$a_{\rho_1} = a_{\rho_2}$

$$\bar{E} = \int_S d\bar{E}_z = \frac{\rho_s}{4\pi\epsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty}\frac{h\rho\, d\rho\, d\phi}{(\rho^2+h^2)^{3/2}}\bar{a}_z = \frac{\rho_s h}{4\pi\epsilon_0}\int_{\rho=0}^{\infty}(\rho^2+h^2)^{-3/2}\rho\, d\rho\,\bar{a}_z$$

$$= \frac{\rho_s h}{\cancel{4\pi}\epsilon_0}\cancel{2\pi}\int_0^{\infty}\left[\rho^2+h^2\right]^{-3/2}\frac{1}{2}d(\rho^2)\bar{a}_z \longrightarrow$$

Since from $x\,dx = \frac{1}{2}d(x^2)$

$$= \frac{\rho_s h}{2\epsilon_0}\left\{-\left[\rho^2+h^2\right]^{-1/2}\right\}_0^{\infty}\bar{a}_z = \frac{\rho_s}{2\epsilon_0}\bar{a}_z$$

let $\rho^2+h^2 = t$

$2\rho d\rho = dt$ $\quad \rho=0\;\; t=h^2$

$\rho d\rho = \frac{dt}{2}$ $\quad \rho=\infty\;\; t=\infty$

$$\bar{E} = \frac{\rho_s h}{2\epsilon_0}\int_{h^2}^{\infty} t^{-3/2}\frac{dt}{2}\bar{a}_z$$

$$= \frac{\rho_s h}{2\epsilon_0}\times\frac{1}{2}\left[\frac{t^{-3/2+1}}{-3/2+1}\right]_{h^2}^{\infty}\bar{a}_z$$

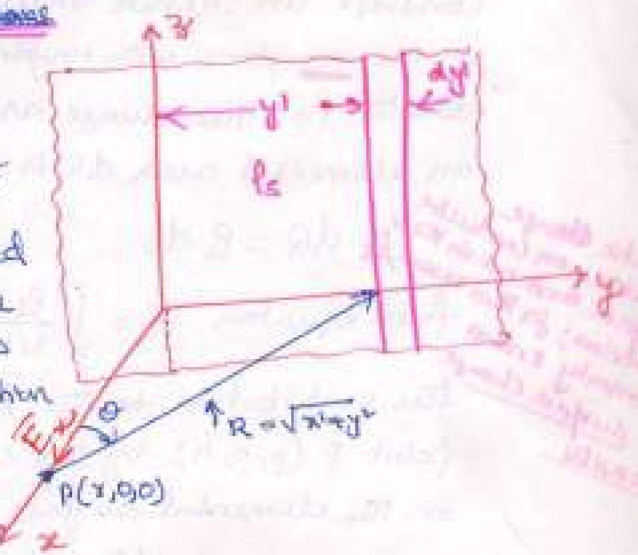
$$= \frac{\rho_s h}{2\epsilon_0}\times\frac{1}{2}\left[\frac{t}{t}\right]_{h^2}^{\infty}\bar{a}_z$$

$$= \frac{\rho_s h}{2\epsilon_0}\times\frac{1}{2}\left[\frac{-1}{h}\right]\bar{a}_z$$

that is, $\bar{E}$ has only z-component if the charge is in xy-plane

The above equation is valid for h>0; for h<0, we would need to replace $\bar{a}_z$ with $-\bar{a}_z$. $\bar{E} = +\frac{\rho_s}{2\epsilon_0}\bar{a}_z$

Alternative Method: (Surface Charge (or) Sheet of Charge)



Let us place a sheet of charge in ~~yz-plane~~ yz-plane and consider symmetry. We first see that the field cannot vary with y or z, and then we see that y & z components arising from differential elements of charge symmetrically located w.r.t the point at which we evaluate the field will cancel. Hence only E_x is present, and this component is a function of x alone.

Let us use the field of infinite line charge by dividing the infinite sheet into differential-width strips.

Then the line charge density or charge per unit length is $\rho_L = \rho_s dy'$ and the distance from this line charge to our general point p on the x-axis is $R=\sqrt{x^2+y^2}$.

The contribution to Ex at p from this differential-width strip is then

$$dE_x = \frac{\rho_s dy'}{2\pi\epsilon_0\sqrt{x^2+y^2}}\cos\theta = \frac{\rho_s}{2\pi\epsilon_0}\frac{x\,dy'}{x^2+y'^2}$$

where $\cos\theta = \frac{x}{\sqrt{x^2+y'^2}}$

from line charge density

Adding the effect of all the strips

$$E_x = \frac{\rho_s}{2\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{x\,dy'}{x^2+y'^2} = \frac{\rho_s}{2\pi\epsilon_0}\tan^{-1}\frac{y'}{x}\Big]_{-\infty}^{\infty}$$

$$\boxed{E_x = \frac{\rho_s}{2\epsilon_0}}$$

$\bar{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}$

$dE_x = \frac{\rho_s dy'}{2\pi\epsilon_0\sqrt{x^2+y^2}}$

$= \frac{\rho_s dy'}{2\pi\epsilon_0\sqrt{x^2+y^2}}\cos\theta$. Component of E along x direction.

If the point p were chosen on the -ve axis

$$\boxed{E_x = \frac{-\rho_s}{2\epsilon_0}}$$

$\frac{y}{x}=\tan\theta$, let $y = x\tan\theta$ then $dy = x\sec^2\theta\, d\theta$

$\frac{x\,dy}{x^2+y^2}$

$x^2+x^2\tan^2\theta$

$\int\frac{x}{x^2+x^2\tan^2\theta}$

$\int\frac{x^2\sec^2\theta}{x^2(1+\tan^2\theta)}d\theta$

$\int d\theta$

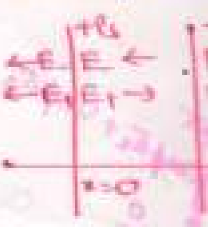
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

The difficulty in sign is usually overcome by the unit vector. $\bar{a}_N$ normal to the sheet directed outwards.

$$\therefore \bar{E} = \frac{\rho_s}{2\epsilon_0}\bar{a}_N$$

* The field is constant in magnitude and direction. It is just as strong a million miles away from the sheet as it is right at the surface.

If a second infinite sheet of charge, a -ve charge density $-\rho_s$, is located plane x=a, we may find the total by adding the contribution of each in the region $x>a$



$\bar{E}_+ = \frac{\rho_s}{2\epsilon_0}\bar{a}_x$

$\bar{E}_- = -\frac{\rho_s}{2\epsilon_0}\bar{a}_x$

$\bar{E} = \bar{E}_+ + \bar{E}_- = 0$

for $x<0$

$\bar{E}_+ = -\frac{\rho_s}{2\epsilon_0}a_x$

$\bar{E}_- = \frac{\rho_s}{2\epsilon_0}\bar{a}_x$

$\therefore \bar{E} = \bar{E}_+ + \bar{E}_- = 0$

and when $0<x<a$

$E_+ = \frac{\rho_s}{2\epsilon_0}\bar{a}_x$ $\quad \bar{E}_- = \frac{\rho_s}{2\epsilon_0}$

$\bar{E} = \bar{E}_+ + \bar{E}_- = \frac{\rho_s}{\epsilon_0}\bar{a}_x$

practical answer for the field between plate of an air capacit

In general, for an infinite sheet of charge

$$\boxed{\overline{E} = \frac{\rho_s}{2\epsilon_0}\overline{a_n}}$$

where $\overline{a_n}$ is a unit vector normal to the sheet.

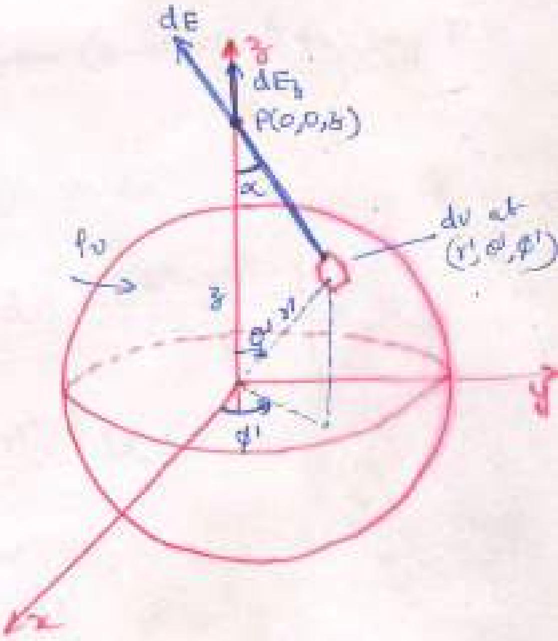
Note! 1. Electric field is normal to the sheet and it is surprisingly independent of the distance between the sheet and the point of observation P. In a parallel plate-capacitor, the electric field existing between the two plates having equal and opposite charges is given by

$$\overline{E} = \frac{\rho_s}{2\epsilon_0}\overline{a_n} + \frac{-\rho_s}{2\epsilon_0}(-\overline{a_n}) = \frac{\rho_s}{\epsilon_0}\overline{a_n}$$

2. The magnitude of $\overline{E}$ is constant every where and given by

$$|E| = \frac{\rho_s}{2\epsilon_0}$$

3. A VOLUME CHARGE



Let us consider a sphere of radius a centered at the origin. Let the volume of the sphere be filled uniformly with a volume-charge density ρ_v (c/m³) as shown in figure.

The charge dQ associated with the elemental volume dv is

$$dQ = \rho_v dv$$

and hence the total charge in a sphere of radius a is

$$Q = \int \rho_v dv = \rho_v \int_v dv = \rho_v \frac{4\pi a^3}{3}$$

The electric field $d\overline{E}$ outside the sphere at P(0,0,z) due to the elementary volume charge is

$$d\overline{E} = \frac{\rho_v dv}{4\pi\epsilon_0 R^2}\overline{a_R}$$

where $\overline{a_R} = \cos\alpha\,\overline{a_z} + \sin\alpha\,\overline{a_\rho}$ owing to symmetry of the charge distribution, the contribution to E_x, or E_y add up to zero. We are left with only E_z, given by

$$E_z = \overline{E}\cdot\overline{a}_z = \int dE\cos\alpha = \frac{\rho_v}{4\pi\epsilon_0}\int\frac{dv\cos\alpha}{R^2} \longrightarrow ①$$

Again, we need to derive expressions for dv, R^2 and $\cos\alpha$

$$dv = r'^2\sin\theta'\,dr'\,d\theta'\,d\phi' \longrightarrow ②$$

applying the cosine rule to the figure

$$R^2 = z^2 + r'^2 - 2zr'\cos\theta'$$

$$r'^2 = z^2 + R^2 - 2zR\cos\alpha$$

It is convenient to evaluate the integral in eq ① in terms of R & r'. Hence we express $\cos\theta'$, $\cos\alpha$ and $\sin\theta' d\theta'$ in terms of R and r', that is

$$\cos\alpha = \frac{z^2 + R^2 - r'^2}{2zR} \longrightarrow ③$$

$$\cos\theta' = \frac{z^2 + r'^2 - R^2}{2zr'} \longrightarrow ④$$

differentiating eq ④ w.r.t θ' and keeping z and r' fixed, we obtain

$$\sin\theta'\,d\theta' = \frac{R\,dR}{zr'} \longrightarrow ⑤$$

As θ' varies from 0 to π, R varies from $(z-r')$ to $(z+r')$ if P is outside the sphere.

Substituting eq ② to eq ⑤ into eq ① yields

$$E_z = \frac{\rho_v}{4\pi\epsilon_0}\int_{\phi'=0}^{2\pi} d\phi' \int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'^2\frac{R\,dR}{zr'}\,dr'\,\frac{z^2+R^2-r'^2}{2zR}\,\frac{1}{R^2}$$

$$= \frac{\rho_v 2\pi}{8\pi\epsilon_0 z^2}\int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'\left[1 + \frac{z^2 - r'^2}{R^2}\right]dR\,dr'$$

$$= \frac{\rho_v\pi}{4\pi\epsilon_0 z^2}\int_0^a r'\left[R - \frac{(z^2 - r'^2)}{R}\right]_{z-r'}^{z+r'} dr'$$

$$= \frac{\rho_v\pi}{2\pi\epsilon_0 z^2}\int_0^a 4r'^2\,dr' = \frac{1}{4\pi\epsilon_0}\frac{1}{z^2}\left(\frac{4}{3}\pi a^3\rho_v\right)$$

$$\text{or}\quad \boxed{\overline{E} = \frac{Q}{4\pi\epsilon_0 z^2}\overline{a}_z}$$

This result is obtained for $\overline{E}$ at $P(0,0,z)$. Owing to the symmetry of the charge distribution, the electric field at $P(r,\theta,\phi)$ is readily obtained from the above equation as $\boxed{\overline{E} = \frac{Q}{4\pi\epsilon_0 r^2}\overline{a}_r}$

which is identical to the electric field at the same point due to point charge Q located at the origin or center of the spherical charge distribution.

④ A line charge located parallel to the y-axis at $x = 2$ and $z = 2$ carries a charge density $\rho_L = 10\pi$ nC/m and two sheet charges at $x = 2$, and $y = -4$ carries charge densities of 20 nC/m² and 15 nC/m² respectively. Calculate the electric field intensity E at (1,1,1) due to the three charge distributions.

$\rho_L = 10\pi$ nC/m at (2,0,2)

$\rho_{s_1} = 20$ nC/m² at $x = 2$

$\rho_{s_2} = 15$ nC/m² at $y = -4$.

1. Field at point (1,1,1) due to line charge is

$$\overline{E}_1 = \frac{\rho_L}{2\pi\epsilon_0 R}\overline{a}_R$$

$$\overline{R} = (1-2)\overline{a}_x + (1-1)\overline{a}_y + (1-2)\overline{a}_z$$

$$= -\overline{a}_x - \overline{a}_z =$$

$$|\overline{R}| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\therefore \overline{E}_L = \frac{10\pi \times 10^{-9}}{2\pi \times 8.854\times 10^{-12} \times \sqrt{2}} \frac{(-\overline{a}_x - \overline{a}_z)}{\sqrt{2}}$$

$$= -90\pi(\overline{a}_x + \overline{a}_z) = -282.7(\overline{a}_x + \overline{a}_z)$$

The field at point $x_1 = 1$ due to sheet charge at $x = 2$ is

$$\overline{E}_2 = \frac{\rho_{s_1}}{2\epsilon_0} -(\overline{a}_x)$$

Since $x > x_1$, the field direction is along the ~~south~~ $-x$ axis.

$$\therefore \overline{E}_2 = \frac{-20\times 10^{-9}}{2\times 8.854\times 10^{-12}}\overline{a}_x = -1130.9\,\overline{a}_x$$

The field at point $y_1 = 1$ due to sheet charge at $y = -3$ is

$$\overline{E}_3 = \frac{\rho_{s_2}}{2\epsilon_0}\overline{a}_y = \frac{15\times 10^{-9}}{2\times 8.854\times 10^{-12}} = 848.22\,\overline{a}_y$$

Since $y < y_1$ the field direction is along the +ve y axis.

∴ Total Electric Field intensity E is

$$\overline{E} = \overline{E}_1 + \overline{E}_2 + \overline{E}_3$$

$$= -282.7(\overline{a}_x + \overline{a}_z) - 1130.9\overline{a}_x + 848.22\overline{a}_y$$

$$E = -1413.6\overline{a}_x + 848.22\overline{a}_y - 282.7\,\overline{a}_z \text{ V/m}.$$

④ A sheet of charge $\rho_s = 2nC/m^2$ is present at the plane $x=3$ in free space, and a line charge $\rho_L = 20nC/m$, is located at $x=1, z=4$ (a) Find the magnitude of the EFI at the origin (b) Find the direction of $\bar{E}$ at $P(4,5,6)$. (c) What is the force per meter length on the line charge?

Sol: The sheet charge is $\rho_s = 2nC/m^2$ at plane $x=3$ and ~~the~~ line charge is $\rho_L = 20nC/m$ at $x=1, z=4$

(a)(i) Electric field intensity at the origin due to surface charge
The surface charge is at $x=3$ plane. Due to surface charge density the electric field intensity (EFI) at origin (0,0,0) is

$$\bar{E}_s = \frac{\rho_s}{2\epsilon_0}\bar{a}_N$$

The unit vector normal to the sheet directed away from it is

$$\bar{a}_N = -\bar{a}_x$$

$$\therefore \bar{E}_s = \frac{\rho_s}{2\epsilon_0}\bar{a}_N = \frac{-2\times10^{-9}\times \overset{4.5}{9}\times10^{9}\,\bar{a}_x}{} = -112.99\bar{a}_x$$

(ii) Electric field intensity at the origin due to line charge

$$R = \sqrt{(0-1)^2+(0-4)^2} = \sqrt{17}$$

$$\bar{E}_L = \frac{\rho_L}{2\pi\epsilon_0 R}\bar{a}_R$$

$$\text{where } \bar{a}_R = \frac{\bar{R}}{|R|} = \frac{(0-1)\bar{a}_x+(0-4)\bar{a}_z}{\sqrt{17}} = \frac{-\bar{a}_x-4\bar{a}_z}{\sqrt{17}}$$

$$\therefore \bar{E}_L = \frac{20\times10^{-9}}{2\pi\epsilon_0\times17}(-\bar{a}_x-4\bar{a}_z) = -21.17\bar{a}_x - 84.67\bar{a}_z$$

∴ Total EFI at the origin is

$$E = E_s + E_L = -112.99\bar{a}_x - 21.17\bar{a}_x - 84.67\bar{a}_z$$

$$= -134.16\bar{a}_x - 84.67\bar{a}_z$$

$$\therefore |E| = 158.65\,V/m.$$

(b) Similarly EFI at $P(4,5,6)$ is

$$E = E_s + E_L$$

$$= +112.99\bar{a}_x + \frac{\rho_L}{2\pi\epsilon_0}\frac{(4-1)a_x+(6-4)a_z}{[(4-1)^2+(6-4)^2]}$$

$$= 112.99\bar{a}_x + 83.14\bar{a}_x + 55.36\bar{a}_z = 196.03\bar{a}_x + 55.36\bar{a}_z$$

$$|E| = 203.7\,V/m.$$

Unit vector in the direction of EFI $a_E = \frac{E}{|E|} = \frac{196.03a_x+55.36a_z}{203.7} = 0.96\bar{a}_x+0$

(c) force per meter length on the line charge.

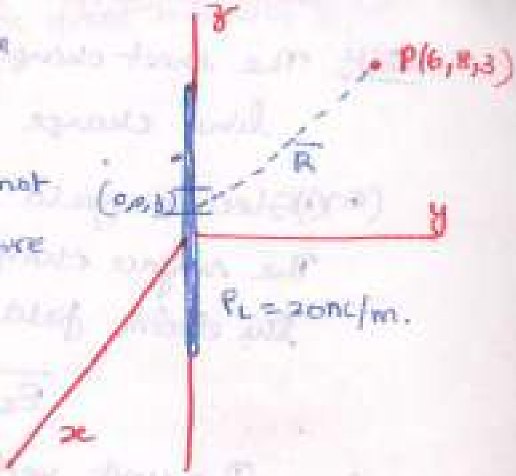
$$F = \rho_L E_s = \rho_L\cdot\frac{\rho_s}{2\epsilon_0}a_N = 20\times10^{-9}(-112.9\bar{a}_x)$$

$$= -2.2598\times10^{-6}\bar{a}_x\,N$$

$$= -2.2598\bar{a}_x\,\mu N$$

Prob A unifom line charge, infinite in extent with $\rho_L = 20nC/m$ lies along the z-axis. Find the $\bar{E}$ at (6,8,3) m.

Sol:- Any point on the line is (0,0,z) since the line lies on the z-axis.



As the line charge is along z-axis, E can not have any components along z-direction. Therefore do not consider while calculating $\bar{R}$

$$\therefore \bar{R} = (6-0)\bar{a}_x + (8-0)\bar{a}_y$$

$$\bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{6\bar{a}_x + 8\bar{a}_y}{\sqrt{6^2+8^2}} = \frac{6\bar{a}_x + 8\bar{a}_y}{10}$$

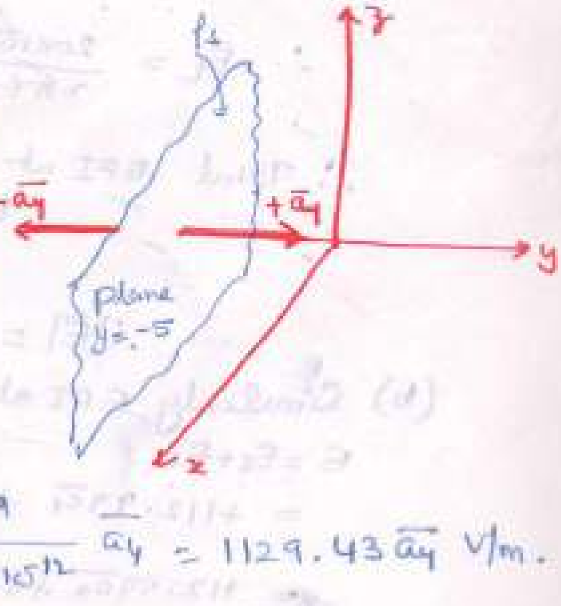
$$= 0.6\bar{a}_x + 0.8\bar{a}_y$$

$$\therefore \bar{E} = \frac{\rho_L}{2\pi\epsilon_0 R}\bar{a}_R = \frac{20\times10^{-9}}{2\pi\times8.854\times10^{-12}\times10}(0.6\bar{a}_x + 0.8\bar{a}_y)$$

$$= 10.7858\,\bar{a}_x + 14.38\,\bar{a}_y \ V/m.$$

Prob - Charge lies in y = -5m plane in the form of an infinite square sheet with a unifom charge density of $\rho_s = 20\,nC/m^2$. Determine E at all the points.

Sol:- The plane y = -5m constant is parallel to xz-plane as shown in figure.



for $y > -5$, the $\bar{E}$ component will be along $+\bar{a}_y$ as normal direction to the plane, y = -5m is $\bar{a}_y$

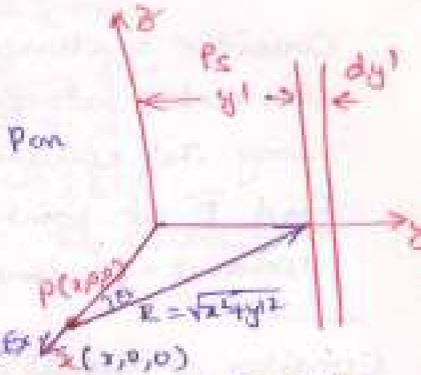
$$\therefore \bar{a}_n = \bar{a}_y$$

$$\therefore \bar{E} = \frac{\rho_s}{2\epsilon_0}\bar{a}_n = \frac{\rho_s}{2\epsilon_0}\bar{a}_y = \frac{20\times10^{-9}}{2\times8.854\times10^{-12}}\bar{a}_y = 1129.43\,\bar{a}_y \ V/m.$$

for $y = < -5$ the component will be along $-\bar{a}_y$ direction, with same magnitude

$$\therefore E = \frac{\rho_s}{2\epsilon_0}(-\bar{a}_y) = -1129.43\,\bar{a}_y \ V/m.$$

5. In the figure, an infinite sheet of charge is in the yz-plane, determine the electric field at a general point P on the x-axis.



Sol:- The line charge per unit length is $\rho_L = \rho_s dy'$

The distance from this line charge to general point P on the x-axis is

$R = |\bar{R}| = \sqrt{x^2+y'^2}$ ~~let~~ $\bar{R} = x\bar{a}_x + y'\bar{a}_y$

The contributing field at point P from this differential width strip is

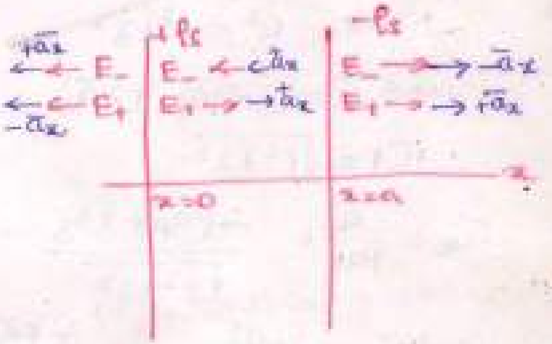
$dE_x = \frac{\rho_s dy'}{2\pi\epsilon_0}$

$$dE_x = \frac{\rho_s dy'}{2\pi\epsilon_0 \sqrt{x^2+y'^2}} \cancel{\cos\theta} = \frac{\rho_s x\, dy'}{2\pi\epsilon_0 (x^2+y'^2)} \qquad a_R = \frac{x\bar{a}_x + y'\bar{a}_y}{\sqrt{x^2+y'^2}}$$

$$\therefore \bar{E}_x = \frac{\rho_s}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{x\, dy'}{(x^2+y'^2)} = \frac{\rho_s}{2\pi\epsilon} \left[\tan^{-1}\left(\frac{y'}{x}\right)\right]_{-\infty}^{\infty} \simeq \frac{\rho}{2\epsilon_0}$$

In general $\bar{E} = \frac{\rho_s}{2\epsilon_0} \bar{a}_N$

6. An infinite sheet of charge $+\rho_s$ is at yz-plane. Another sheet of charge $-\rho_s$ is located in the plane $x = a$. Determine field at $x > a$, $x < 0$, and $0 < x < a$.



Sol:- In the region $x > a$ $E_+ = \frac{\rho_s}{2\epsilon_0}\bar{a}_x$

$E_- = -\frac{\rho_s}{2\epsilon_0}\bar{a}_x$

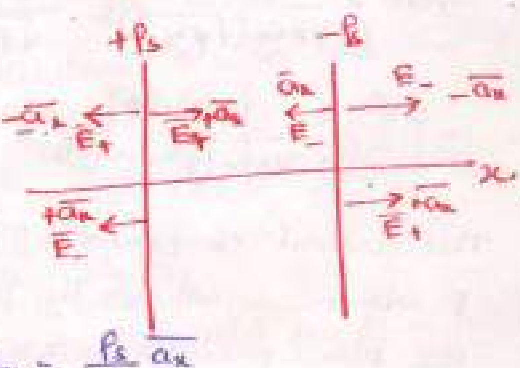
The total field is $E = E_+ + E_- = 0$.

for $x < 0$

$E_+ = \frac{-\rho_s}{2\epsilon_0}\bar{a}_x$

$E_- = \frac{\rho_s}{2\epsilon_0}\bar{a}_x$

$\therefore E = E_+ + E_- = 0$



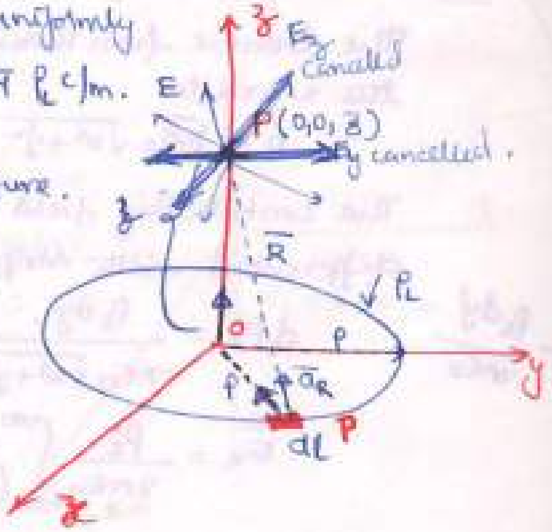
for $0 < x < a$ $E_+ = \frac{\rho_s}{2\epsilon_0}\bar{a}_x$

$E_- = \frac{\rho_s}{2\epsilon_0}\bar{a}_x$

$\therefore$ total field $E = E_+ + E_- = \frac{2\rho_s}{2\epsilon_0} a_x = \frac{\rho_s}{\epsilon_0}\bar{a}_x$

Electric Field due to Charged Circular Ring.

Consider a charged circular ring of radius 'ρ' placed in xy plane with centre at origin, carrying a charge uniformly along its circumference. The charge density ρ_L C/m. Find $\overline{E}$ at point P at a perpendicular distance 'z' from the ring as shown in figure.



Consider a small differential length dl on the ring. The charge on it is dQ.

$$\therefore dQ = \rho_L \, dl$$

$$\therefore d\overline{E} = \frac{\rho_L \, dl}{4\pi\epsilon_0 R^2} \overline{a}_R$$

where R – distance of point P from dl.

Consider the cylindrical coordinate system.

For dl we are moving in φ direction

where $dl = \rho d\phi$

Now $R^2 = \rho^2 + z^2$

and $\overline{R} = -\rho \overline{a}_\rho + z\overline{a}_z$

$$\therefore |\overline{R}| = \sqrt{\rho^2 + z^2}$$

$$\overline{a}_R = \frac{\overline{R}}{|\overline{R}|} = \frac{-\rho \overline{a}_\rho + z\overline{a}_z}{\sqrt{\rho^2+z^2}}$$

$$\therefore d\overline{E} = \frac{\rho_L \, dl}{4\pi\epsilon_0 (\sqrt{\rho^2+z^2})^2} \frac{-\rho\overline{a}_\rho + z\overline{a}_z}{\sqrt{\rho^2+z^2}}$$

$$d\overline{E} = \frac{\rho_L (\rho \, d\phi)}{4\pi\epsilon_0 (\rho^2+z^2)^{3/2}} (-\rho \overline{a}_\rho + z\overline{a}_z)$$

The radial components of $\overline{E}$ at point P will be symmetrically placed in the plane parallel to xy plane and are going to cancel each other.

$$\therefore d\overline{E} = \frac{\rho_L (\rho \, d\phi)}{4\pi\epsilon_0 (\rho^2+z^2)^{3/2}} z\overline{a}_z$$

$$\therefore \overline{E} = \int_{\phi=0}^{2\pi} \frac{\rho_L \rho \, d\phi}{4\pi\epsilon_0 (\rho^2+z^2)^{3/2}} z\overline{a}_z$$

$$= \frac{\rho_L \rho}{4\pi\epsilon_0 (\rho^2+z^2)^{3/2}} z\overline{a}_z [\phi]_0^{2\pi}$$

$$\therefore \overline{E} = \frac{\rho_L \rho z}{\cancel{4}\pi\epsilon_0 (\rho^2+z^2)^{3/2}} \overline{a}_z \ (2\pi)$$

$$\boxed{E = \frac{\rho_L \rho z}{2\epsilon_0 (\rho^2+z^2)^{3/2}} \overline{a}_z}$$

is $\overline{E}$ at point P(0,0,z) due to the circular ring of radius ρ placed in xy-plane.

$$\frac{\rho_L}{2\epsilon_0 (1+1)^{3/2}} a_z$$

$$= \frac{\rho_L}{2\epsilon_0 \, 2^{3/2}} \overline{a}_z$$

Example! A circular disk of radius a is uniformly charged with ρ_s C/m². The disk lies on the $z = 0$ plane with its axis along the z-axis.

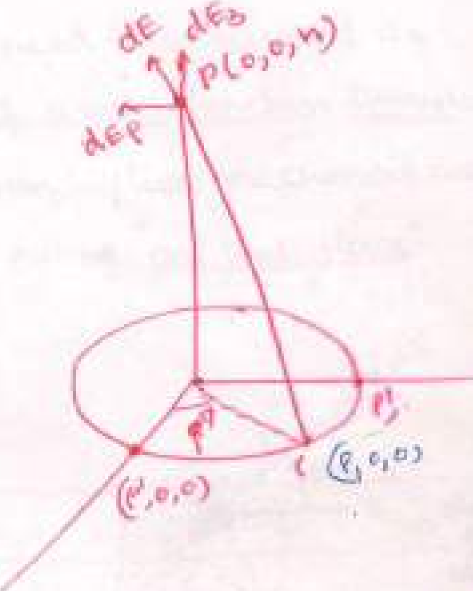
(a) Show that at point (0,0,h) $E = \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{h}{[h^2+a^2]^{1/2}}\right]\overline{a_z}$.

Sol!- Uniform charge density ρ_s in circular region $\rho < a$, $z = 0$

Selecting a small segment of $\rho' d\rho' d\phi$ on the disk. The electric field intensity at point (0,0,h) is

$$E = \int_0^{2\pi}\int_0^{a} \frac{\rho_s \rho' d\rho' d\phi}{4\pi\epsilon_0} \times \frac{[(0-\rho')\overline{a_\rho} + (h-0)\overline{a_z}]}{[\rho'^2 + h^2]^{3/2}}$$

The components are calculated separately



$$E_\rho = \int_0^a 2\pi \frac{\rho_s}{4\pi\epsilon_0} \frac{\rho'^2 d\rho'}{(\rho'^2+h^2)^{3/2}}$$

$$= -\frac{\rho_s}{\epsilon_0}\int_0^a \frac{\rho'^2 d\rho'}{(\rho'^2+h^2)^{3/2}}$$

$$E_z = \int_0^a 2\pi \frac{\rho_s h}{4\pi\epsilon_0} \frac{\rho' d\rho'}{(\rho'^2+h^2)^{3/2}}$$

$$= \frac{\rho_s h}{2\epsilon_0} \times \left[-\frac{1}{\sqrt{\rho'^2+h^2}}\right]_0^a$$

$$\overline{E} = \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{h}{\sqrt{a^2+h^2}}\right]\overline{a_z}$$

(b) From this, derive the $\overline{E}$ field due to an infinite sheet of charge on the $z = 0$ plane.

Sol! For infinite sheet the radius of the disk $a = \infty$ and the unit vector along the z-axis

~~$\therefore E_z = \int_0^\infty 2\pi \frac{\rho_s h}{4\pi\epsilon_0}$~~ $\rho' d$ $\overline{E_z} = \lim_{a\to\infty} \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{h}{\sqrt{a^2+h^2}}\right]\overline{a_z}$

$$E_z = \frac{\rho_s}{2\epsilon_0}\overline{a_z}$$

(c) If $a << h$, show that $\overline{E}$ is similar to the field due to a point charge.

~~$E_\rho = \int_0^a 2\pi \frac{\rho_s}{4\pi\epsilon_0}$~~

~~$\overline{E} = \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{h}{\sqrt{h^2}}\right]\overline{a_z}$~~ ~~$E_z = \int_0^a 2\pi$~~

~~$= \frac{\rho_s}{2\epsilon_0}\{1 - 1$~~

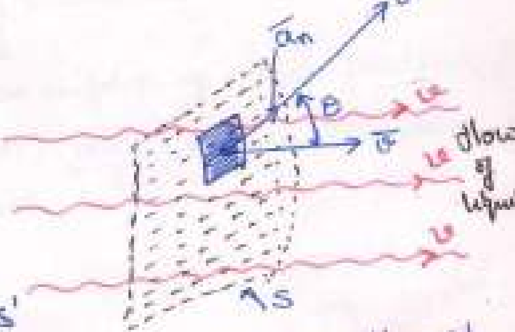
GAUSS'S LAW

There is an important relation between the vector $\overline{E}$ in any electrostatic field and the static charge (Q) producing it. It is the consequence of the mathematical form of electric field strength of a point charge, and is known as Gauss' Law. Among other applications, Gauss' law enables a simple evaluation of the electric field in some simple but important cases.

To understand Gauss' law, we first need to understand an important mathematical concept, the flux of a vector function through a surface. The word "flux" originates from fluid mechanics and comes from a latin word "fluxus" which means "one that flows".

The Concept of Flux

Consider a uniform flow of liquid of velocity $\overline{v}$ that is a function of coordinates but not of time. Imagine a net so fine that it does not disturb the flow of the liquid it is placed in. Let the surface of the net be 'S'



a finite net in a flow of liquid can be approximat by a large numbes of flat surface elements such as $d\overline{S}$

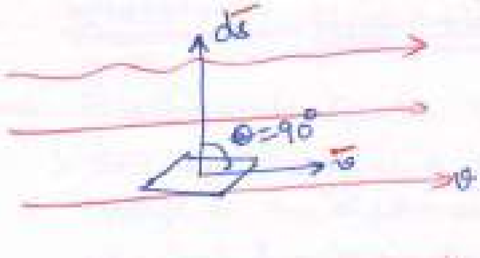
* We wish to determine the amount of the liquid that passes through the net (Through S) in one second.

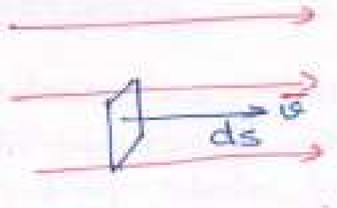
We can subdivide the surface S into a large number of small flat surface elements dS. Obviously, the total amount of liquid passing through the net is obtained as a sum of the small amounts passing through all of the small elements.

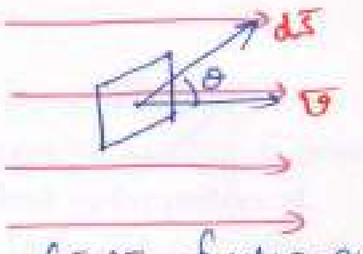
Consider a small flat surface element shown in the figure. The vect $\overline{a}_n$ denotes a unit vector normal to the element, and θ is the angle between this unit vector and local velocity $\overline{v}$ of the fluid.

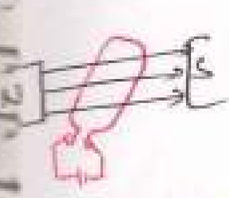
It is evident that if the velocity $\overline{v}$ is tangential to the element, there is not flow of fluid through it.

Therefore, only the component of velocity normal to the element contribu the flow liquid through the element.









$\oint_s \bar{v}.d\bar{s} = 0$
$\theta = 90°$

$\oint_s \bar{v}.d\bar{s} = v ds$
$\theta = 0$

$\oint_s \bar{v} d\bar{s} = \oint_c v ds \cos\theta$

In one second, the fluid at that point moves by a distance normal to ds equal to $v\cos\theta$. The quantity of fluid that passes through ds in one second is therefore $v\cos\theta\, ds$.

The quantity of fluid that passes through S in one second is a sum of all these infinitely small flows (partial flows). It is therefore an integral

Fluid flow through S in one second $= \int_S \bar{v}\cos\theta\, ds$

The expression under the integral sign has a form of a dot product, but although v is the magnitude of a vector, ds is not. If however, we define a vector surface element $d\bar{s}$ as
scalar

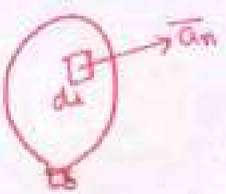
$$d\bar{s} = ds\,\bar{a}_n$$

the above equation can be written as

Fluid flow through S in one second $= \int_S \bar{v}.d\bar{s}$

is known as flux of vector v through the surface S.

It is evident that the surface S can be a closed surface, therefore the above equation can be written as



Flux of $\bar{E} = \oint_S \bar{E}.d\bar{s}$

$$\text{flux of } v = \oint_S \bar{v}.d\bar{s}$$

It is evident that the concept of flux can be used in connection with any vector function, not necessarily the velocity.

The flux of a vector function through a closed surface is a very important concept in the theory of electromagnetic fields. It is a convention to adopt the unit vector $\bar{a}_n$ normal to a closed surface to be directed from the surface outward.

~~UNIT-I: ELECTROSTATICS~~

~~Syllabus: Electrostatic Fields – Coulomb's law – Electric Field Intensity (EFI) – EFI due to a line & a surface charge – Workdone in moving a poi charge in an electrostatic field – Electric Potential – Properties of potential function – Potential gradient – Gauss law – Application Gauss's law – Maxwell's First law, div D = ρ_v~~

~~Electrostatics is a facinating subject that has grown up in diverse area of application. Electric power transmission, X-ray machines, and lightnin protection are associated with strong electric fields and will require a knowledge of electrostatics to understand and design suitable equipment.~~

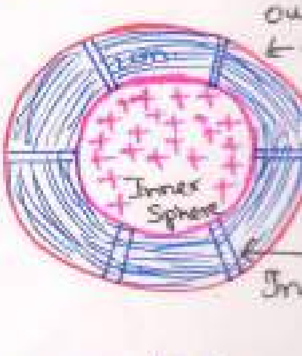
~~We begin our study of electrostatics by investigating the two fundam laws governing electrostatic fields: (1) Coulomb's law, and (2) Gauss's la Both of these laws are based on experimental studies, and they are interdependent.~~

Electric Flux Density

In 1837, Michael Faraday, Director, Royal Society ~~of~~ in London, had experimented with a pair of concentric spheres. The outer one consisting of two hemispheres that could be firmly clamped together.

His experiment, consisted essentially of the following steps:

1. With the equipment dismantled, the inner sphere was given a known positive charge
2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer sphere was separated carefully, using tools made of insulating material in order to not to disturb the induced charge and the negative induced charge on each hemisphere was mea



UNIT-I ELECTROSTATICS

Syllabus: Electrostatic Fields - Coulomb's Law - Electric Field Intensity (EFI) - EFI due to a line & a surface charge - Work done in moving a point charge in an electrostatic field - Electric Potential - properties of potential function - Potential gradient - Gauss's law - Application of Gauss's law - Maxwell's first law, div D = ρ_v

Electrostatics is a fascinating subject that has grown up in diverse areas of application. Electric power transmission, X-ray machines, and lightning protection are associated with strong electric fields and will require a knowledge of electrostatics to understand and design suitable equipment.

We begin our study of electrostatics by investigating the two fundamental laws governing electrostatic fields: (1) Coulomb's law and (2) Gauss's law. Both of these laws are based on experimental studies and they are interdependent.

Electric Flux $\Psi = \oint \bar{E}.d\bar{s}$

Electric Flux Density:

Flux of $\bar{E} = \oint \bar{E}.d\bar{s} = \frac{Q_{total\,inc}}{\epsilon_0}$ V-m

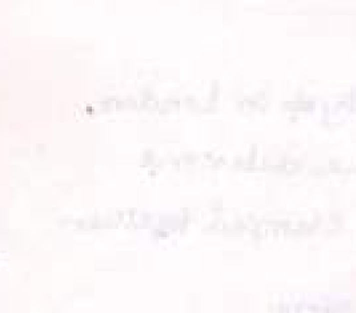
$= E\oint_s ds = E4\pi r^2$

$= \frac{Q}{4\pi\epsilon_0 r^2}4\pi r^2 = \frac{Q}{\epsilon_0}$

In 1837, Michael Faraday, Director of Royal Society in London, had experimented with a pair of concentric spheres. The apparatus consists of two hemispheres that could be firmly clamped together.

The experimental procedure essentially the following steps:

1. With the equipment dismantled, the inner sphere was given a known positive charge.



2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer sphere was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

Faraday found that the total charge on the outer sphere was equal in magnitude to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres. He concluded that there was some sort of "displacement" from the inner sphere to the outer sphere which was independent of the medium, and we now refers to this flux as displacement, displacement flux or simply electric flux(ψ)

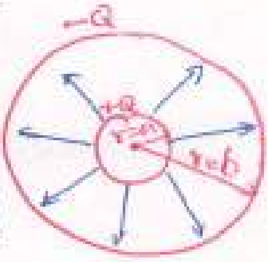
If electric flux is denoted by ψ (psi) and the total charge on the inner sphere by Q, then

Electric flux, $\psi = Q$ coulombs

also called displacement flux.

If the charge on a body is ±Q C, the total no of flux lines (force lines) originating or terminating on it is also Q. The total no of line = a flux.

The pattern of electric flux ψ extending from the inner sphere to the outer sphere are indicated by the symmetrically distributed ~~steam~~ streamlines drawn radially from ~~the~~ one sphere to the other.



At the surface of the inner sphere, ψ coulombs of electric flux are produced by the charge $Q(=\psi)$ coulombs distributed uniformly over a surface having an area of $4\pi a^2\ m^2$. The density of the flux at this surface is $\psi/4\pi a^2$ or $Q/4\pi a^2\ C/m^2$.

Electric flux density ($\overline{D}$) is measured in coulombs per sq meter (lines per square meter).

Referring to figure. The electric flux density is in the radial direction and has a value of

$$\overline{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\overline{a_r} \quad \text{(inner sphere)}$$

$$\overline{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\overline{a}_r \quad \text{(outer sphere)}$$

and at a radial distance r, where $a \le r \le b$

$$\overline{D} = \frac{Q}{4\pi r^2}\overline{a_r}$$

If we now let the inner sphere become smaller & smaller, while still retaining a charge of Q, it becomes a point charge in the limit, but electric flux density at a point r meters from the point charge is still given by $\overline{D} = \frac{Q}{4\pi r^2}\overline{a_r}$

The above equation should be compared with radial electric field Intensity of a point charge in free space, $E = \frac{Q}{4\pi\epsilon_0 r^2}\bar{a}_r$

$$\epsilon_0 E = \frac{Q}{4\pi r^2}\bar{a}_r = \bar{D}$$

$$\therefore \quad D = \epsilon_0 E \quad \text{(free space only).}$$

For volume charge distribution in free space

$$E = \int_{vol} \frac{\rho_v dv}{4\pi\epsilon_0 R^2}\bar{a}_R$$

leads to

$$D = \epsilon_0 E = \int_v \frac{\rho_v dv}{4\pi R^2}\bar{a}_R$$

ψ in terms of D is

$$\psi = \int_s D.ds$$

It appears that all the formulas derived for $\bar{E}$ from coulomb's law can be used to in calculating D, except that we have to multiply those eq by

Electric Flux Density for Various Charge Distributions

1. Line charge

$$E = \frac{\rho_L}{2\pi\epsilon_0 r}\bar{a}_r$$

$$D = \epsilon_0\bar{E}$$

$$\therefore \bar{D} = \frac{\rho_L}{2\pi r}\bar{a}_r \quad \text{—— Infinite line charge.}$$

2. Surface charge

$$\bar{E} = \frac{\rho_s}{2\epsilon_0}a_n$$

$$\bar{D} = \epsilon_0\bar{E}$$

$$\bar{D} = \frac{\rho_s}{2}\bar{a}_n \quad \text{—— Infinite sheet charge.}$$

3. Volume charge

$$\bar{E} = \frac{\int_v \rho_v dv}{4\pi\epsilon_0 r^2}\bar{a}_r$$

$$\bar{D} = \epsilon_0 E$$

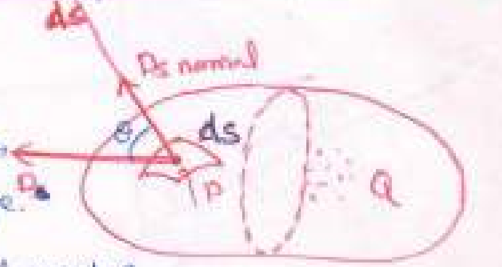
$$\therefore \bar{D} = \frac{\int_v \rho_v dv}{4\pi r^2}\bar{a}_r$$

GAUSS'S LAW ~~Applications~~

Gauss's law constitutes one of the fundamental laws of electromagnetism

The generalizations of Faraday's experiments lead to the following stateme which is known as Gauss's law

Def:- The electric flux passing through any closed surface is equal to the total charge enclosed by that surface



Let us imagine a distribution of charge, shown as a cloud of point charges in figure, surrounded by a closed surface of any shape.

At any point P consider an incremental element of surface ds and let $\overline{D_s}$ make an angle θ with $d\overline{s}$ as shown in figure. The flux crossing ds is then the product of the norma component of $\overline{D_s}$ and $d\overline{s}$

$$d\psi = \text{flux crossing } ds = D_{s,\text{norm}}\, ds = D_s \cos\theta\, ds = \overline{D_s}\cdot d\overline{s}$$

The total flux passing through the closed surface is obtained by adding the differential contributions crossing each surface element $d\overline{s}$

$$\psi = \int d\psi = \oint_{\substack{\text{closed}\\ \text{surface}}} \overline{D_s}\cdot d\overline{s} = \text{total charge enclosed } Q = \int_v \rho_v dv$$

Qenclosed in a surface

Q can be several charges $Q = \Sigma Q_n$

or line charge $Q = \int \rho_L dL$

surface " $Q = \int_s \rho_s ds$

volume charge $Q = \int_{vol} \rho_v dv$

$$\psi = \oint \overline{D} d\overline{s} = Q$$

$$\boxed{\oint_s \overline{D_s}\, d\overline{s} = \int_v \rho_v dv = \int_v \nabla . D\, dv}$$ [by applying diverge theorem]

A mathematical statement meaning simply that the total electric flu through any closed surface is equal to the charge enclosed.

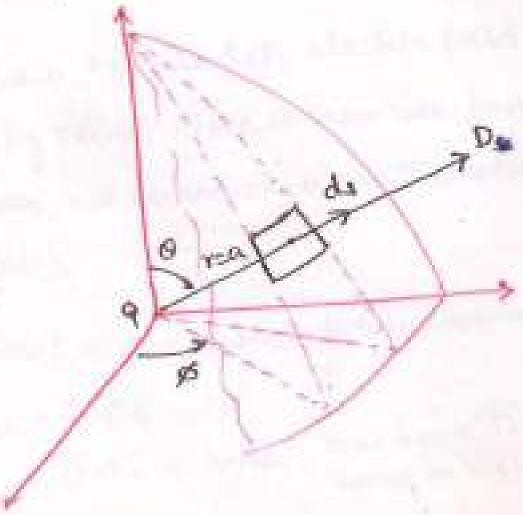
* Gauss' law is a very simple and important consequence of the mathematical form of expression of the vector E of a point charge (i.e of the electric field strength vect Coulomb's law). It states that flux of the electric field equals the total through any closed surface in the electrostatic field equals the total charge enclosed by the surface, divided by ϵ_0. $\oint_s E.ds = \frac{Q_{total\,ins}}{\epsilon_0}$ (v.m

Basically Gauss' law is a relationship between the sources inside a closed surface and the field they produce over this entire surface.

Application of Gauss's Law:

To illustrate the application of Gauss's law, let us check the results of Faraday's experiment by placing a point charge Q at origin of a spherical coordinate system, and by choosing our closed surface as a sphere of radius 'a'. The electric field intensity of the point charge has been found to be

$$\overline{E} = \frac{Q_1}{4\pi\epsilon_0 r^2}\overline{a}_r$$



and since

$$\overline{D} = \epsilon_0\overline{E}$$

$$\therefore \overline{E} = \frac{Q}{4\pi r^2}\overline{a}_r$$

At the surface of the sphere

$$\overline{D_s} = \frac{Q}{4\pi a^2}\overline{a}_r$$

The differential elemental area on a spherical surface is

$$ds = r^2 \sin\theta \, d\theta \, d\phi \quad = a^2 \sin\theta \, d\theta \, d\phi$$

$$\text{or} \qquad d\overline{s} = a^2 \sin\theta \, d\theta \, d\phi \, \overline{a}_r$$

$$\therefore \overline{D_s} \cdot d\overline{s} = \frac{Q}{4\pi a^2} a^2 \sin\theta \, d\theta \, d\phi \, \overline{a}_r \cdot \overline{a}_r = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

leading to the closed surface integral

$$\oint_s \overline{D_s} \cdot d\overline{s} = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=\phi}^{\theta=\pi} \frac{Q}{4\pi} \sin\theta \, d\phi$$

Limits are chose such that integration is carr over the entire surface the sphere once

$$= \int_{\phi=0}^{2\pi} \frac{Q}{4\pi} (-\cos\theta)_0^\pi \, d\phi$$

$$\oint_s \overline{D} \cdot d\overline{s} = \int_0^{2\pi} \frac{Q}{2\pi} d\phi = Q.$$

and we obtain a result showing that Q coulombs of electric flu are crossing the surface

The solution is easy if we are able to choose a closed surface
~~So that~~ which satisfies two conditions:

1. $\overline{D}$ is everywhere either normal or tangential to the closed surface, so that $\overline{D}.d\overline{s}$ becomes either $D\,ds$ or zero, respectively.
2. On that portion of the closed surface for which $\overline{D}.d\overline{s}$ is not zero, D = constant.

The procedure for applying Gauss's Law to calculate electric field involves first knowing whether symmetry exists. Once it has bee found that symmetric charge distribution exists, we construct a mathematica closed surface (known as a Gaussian Surface).

The surface is chosen that $\overline{D}$ is normal or tangential to Gaussian Surfa

When $\overline{D}$ is tangential to the surface $\overline{D}.d\overline{s} = 0$

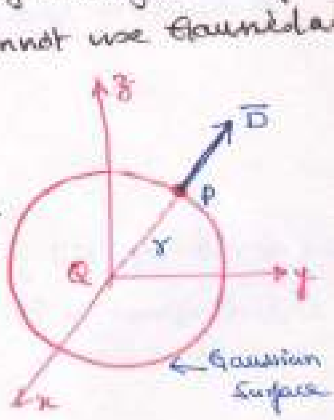
When $\overline{D}$ is normal to the surface $\overline{D}d\overline{s} = D\,ds$ because D is const. on the surf

* Thus we must choose a surface that has some of the symmetry exhibited by the charge distribution.

We shall now apply these basics ideas to the following cases:

1. point Charge:
2. Infinite Line Charge
3. Infinite Sheet of Charge
4. Uniformly charged Sphere.

Note: The application of Gauss's law depends on symmetry, and if we can not show that symmetry exists then we cannot use Gauss's la to obtain a solution.



1. Point Charge:

Suppose a point charge Q is located at the origin. To determine $\overline{D}$ at a point P, it is easy to see that choosing a spherical surface containing P will satisfies symmetry condition. Thus a spherical surface centered at the origin is the Gaussian surface in this case as shown in figure.

Since $\overline{D}$ is everywhere normal to the Gaussian Surface that is, $\overline{D} = D_r \overline{a}_r$, applying Gauss's Law ($\Psi = Q_{enclosed}$) gives

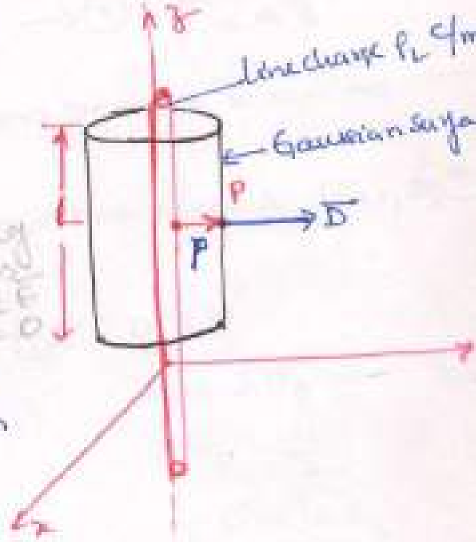
$$Q = \oint_S \overline{D}.d\overline{s} = D_r \oint_S ds = D_r\, 4\pi r^2$$

where $\oint ds = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi = 4\pi r^2$ is the surface area of the Gaussian surface. Thus

$$D = \frac{Q}{4\pi r^2}\,\bar{a}_r = \epsilon_0 \underline{E}$$

2. Infinite Line Charge (Uniform Line Charge)

Suppose the infinite line of uniform charge ρ_L C/m lies along the z-axis. To determine $\bar{D}$ at a point P, we chose a cylindrical surface containing P to satisfy the symmetry condition as shown in figure



The electric flux density $\bar{D}$ is constant and everywhere normal to the cylindrical Gaussian surface, that is

$$\bar{D} = D_\rho \bar{a}_\rho$$

Applying Gauss's law

$$Q = \oint_{cyl} \bar{D}\cdot d\bar{s} = D\int_{sides} ds + \overbrace{0\int_{top} ds + 0\int_{bottom} ds}^{\text{Since } \bar{E} \text{ is zero along z-axis } \bar{D} = \epsilon\bar{E} = 0}$$

$$= D\int_{z=0}^{L}\int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = D \, 2\pi\rho L$$

(or) $\rho_L L = Q = \oint_S \bar{D}\cdot d\bar{s} = D_\rho \oint_S ds$

$= D_\rho \, 2\pi\rho L$

where $\oint ds = 2\pi\rho L$ is the surface area.

∴ and obtain

$$D = D_\rho = \frac{Q}{2\pi\rho L}$$

In terms of the charge density ρ_L, the total charge enclosed is

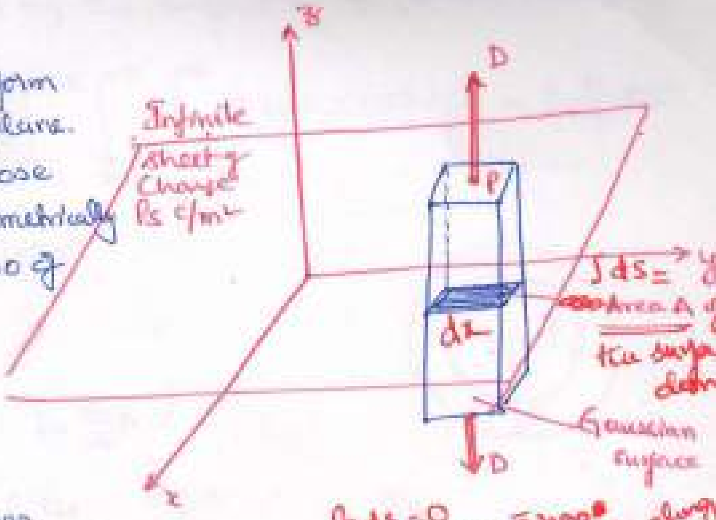
$$Q = \rho_L L$$

giving $$D_\rho = \frac{\rho_L}{2\pi\rho}$$

Note! $\int \bar{D}\cdot d\bar{s}$ evaluated on top & bottom surfaces of the cylinder is zer since $\bar{D}$ has no z-component; that means $\bar{D}$ is tangential to those surfaces. Thus

$$\bar{D} = \frac{\rho_L}{2\pi\rho}\bar{a}_\rho = \epsilon_0\bar{E}$$

3. Infinite Sheet of Charge



Consider an infinite sheet of uniform charge ρ_s C/m² lying on the $z=0$ plane. To determine $\bar{D}$ at point P, we choose a rectangular box that is cut symmetrically by the sheet of charge and has two of its faces parallel to the sheet.

As $\bar{D}$ is normal to the sheet

$$\bar{D} = D_z\bar{a}_z$$

and applying of Gauss's Law gives

$$\rho_s\int_s ds = Q = \oint_S \bar{D}\cdot d\bar{s} = D_z\left[\int_{top} ds + \int_{bottom} ds\right] \longrightarrow ①$$

as $\oint_{side} D\, ds = 0$ since $\bar{D}$ has no component along x & y.

$\bar{D}\cdot d\bar{s}$ evaluated on the sides of the box is zero because $\bar{D}$ has no components along $\bar{a}_x$ and $\bar{a}_y$.

If the top and bottom area of the box, each has area A, then eq(

becomes

$$\rho_s A = D_z(A+A) \Rightarrow \rho_s \cancel{A} = D_z 2\cancel{A}$$

$$\rho_s = 2D_z$$

$$D_z = \frac{\rho_s}{2}$$

$$\therefore \bar{D} = \frac{\rho_s}{2}\bar{a}_z$$

and thus $\bar{D} = \frac{\rho_s}{2}\bar{a}_z = \epsilon_0\bar{E}$

or $\bar{E} = \frac{\bar{D}}{\epsilon_0} = \frac{\rho_s}{2\epsilon_0}\bar{a}_z$

4. Uniformly Charged Sphere

Gaussian surface for a uniform charged spheres $r \geq a$, & $r \leq$

Consider a sphere of radius a with a uniform charge ρ_0 C/m³. To determine $\bar{D}$ everywhere, we construct Gaussian surfaces for cases $r \leq a$ & $r \geq a$ separately.

Since the charge has spherical symmetry, it is obvious that a spherical surface is an appropriate Gaussian surface.

⊛ For $r \leq a$ the total charge enclosed by the spherical surface of radiu r is,

$$Q_{encl} = \int_v \rho_v dv = \rho_0\int_v dv = \rho_0\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{r} r^2\sin\theta\, dr\, d\theta\, d\phi = \rho_0\frac{4}{3}$$

and $\psi = \oint_S \bar{D}\cdot d\bar{s} = D_r\oint_S ds = D_r\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} r^2\sin\theta\, d\theta\, d\phi = D_r 4\pi r^2$

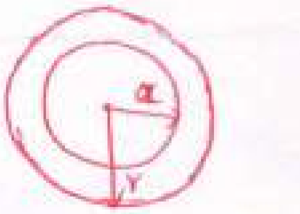
Hence $\psi = Q_{encl}$ gives $D_r 4\pi r^2 = \frac{4\pi r^3}{3}\rho_0$

or $\bar{D} = \frac{r}{3}\rho_0\bar{a}_r \quad 0 < r \leq a$

For $r \geq a$, the charge enclosed by the surface is the entire charge in this case, that is,

$$Q_{encl} = \int_v \rho_0 dv = \rho_0 \int_v dv = \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi = \rho_0 \frac{4}{3} \pi a^3$$

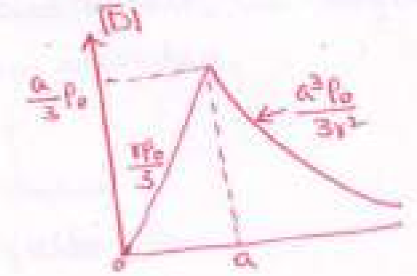
while $\psi = \oint_s \bar{D} \cdot d\bar{s} = D_r 4\pi r^2$



Hence, $\psi = Q_{encl}$ gives

$$D_r 4\pi r^2 = \rho_0 \frac{4}{3} \pi a^3$$

$$\bar{D} = \frac{a^3}{3r^2} \rho_0 \bar{a}_r \quad r \geq a$$



∴ $\bar{D}$ everywhere is given by

$$\bar{D} = \begin{cases} \frac{r}{3} \rho_0 \bar{a}_r & 0 < r \leq a \\ \frac{a^3}{3r^2} \rho_0 \bar{a}_r & r \geq a \end{cases}$$

Note:- From the above applications, the ability to take $\bar{D}$ out g the integral sign is the key to finding $\bar{D}$ using Gauss's law. In other words, $\bar{D}$ must be constant on the Gaussian surface.

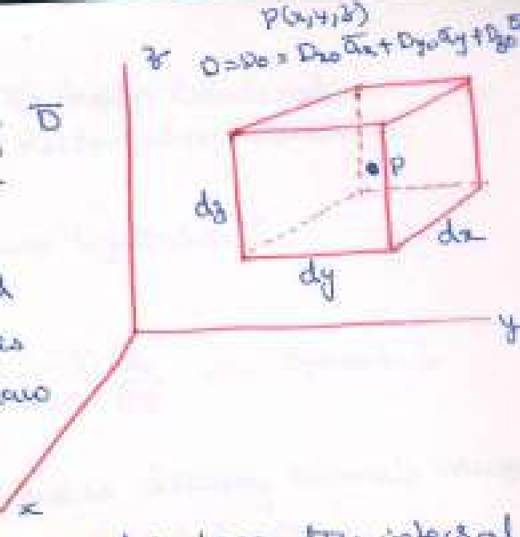
5. Differential Volume Element (Asymmetrical Charge Distribution)

We are now going to apply the method of Gauss'law to a slightly different type of problem - one which does not possess any symmetry at all. At first glance it might seen that our case is hopeless, for without symmetry a simple gaussian surface cannot be chosen such that the normal component of $\bar{D}$ is constant or zero everywhere on the surfac. Without such a surface, the integral cannot be evaluated. There is only one way to circumvent these difficulties, and that is to choose such a very small closed surface that $\bar{D}$ ~~also~~ is almost constant over the surface, and a small change in $\bar{D}$ may be adequately represented by using first two terms of Taylor's-series expansion of $\bar{D}$. The result will become more nearly correct as the volume enclosed by the gaussian surface decreases, and we intend eventually to allow this volume to approach zero.

This example also differs from the preceding ones in that we shall not obtain the value of $\bar{D}$ as our answer, but will instead receive some extrem valuable information about the way $\bar{D}$ varies in the region of our small surf. This leads directly to one of Maxwell's four equations, which are basic to EM Theory.

Let us consider any point P, located by a rectangular coordinate system. The value of $\overline{D}$ at point P may be expressed in rectangular components $\overline{D}_0 = D_{x0}\overline{a}_x + D_{y0}\overline{a}_y + D_{z0}\overline{a}_z$.

We choose as our closed surface the small rectangular box, centered at P, having sides of length dx, dy, and dz and apply Gauss' law



$$\oint_s \overline{D}.d\overline{s} = Q$$

Inorder to evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face.

$$\oint_s \overline{D}.d\overline{s} = \int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom}$$

By Taylor's series expansion & recombination, yields

$$\oint_s \overline{D}.d\overline{s} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) dx\, dy\, dz$$

or

$$\oint_s \overline{D}.d\overline{s} = Q = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) dv \longrightarrow ①$$

This expression is an approximation which becomes better as dv becomes smaller, ~~and in the~~

$$\boxed{\text{Charge enclosed in volume } \Delta v = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \times \text{volume } \Delta v}$$

DIVERGENCE

We shall now obtain an exact relationship from ① by allowi the volume element Δv to shrink to zero.

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \frac{\oint_s \overline{D}.d\overline{s}}{\Delta v} = \frac{Q}{\Delta v}$$

or as a limit

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint \overline{D}.d\overline{s}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v \text{ volum charg Den}$$

Theorem: The divergence of the ~~vector~~ electric flux density $\overline{D}$ is the outfl of flux from a small closed surface per unit volume a the volume thrinks to zero.

$$\text{Divergence of } \overline{D} = \text{div } \overline{D} = \lim_{\Delta v \to 0} \frac{\oint_s \overline{D}.d\overline{s}}{\Delta v}$$

$$\nabla . D = \text{div } \bar{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$ → Kartesian Coordinates
Rectangular Coordinates

$$\nabla . D = \text{div } \bar{D} = \frac{1}{\rho}\frac{\partial(\rho D_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$ → cylindrical

$$\nabla . D = \text{div } \bar{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$ — Spherical.

Divergence merely tells us how much flux is leaving a small volume on a per-unit volume basis; no direction is associated with it.

DIVERGENCE THEOREM

This theorem applies to any vector field for which the appropriate partial derivatives exist, although it is easier for us to develop it for the electric flux density.

Gauss Law $\oint_S \bar{D} . d\bar{S} = Q$

and letting $Q = \int_v \rho_v dv$

and then replacing ρ_v by its equal, $\nabla . \bar{D} = \rho_v$

we have

$$\oint_S \bar{D} . d\bar{S} = Q = \int_{vol} \rho_v dv = \int_{vol} \nabla . \bar{D} dv$$

$$\Rightarrow \boxed{\oint_S \bar{D} . d\bar{S} = \int_{vol} \nabla . \bar{D} dv}$$ Divergence Theorem

Statement: The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

MAXWELL'S FIRST EQUATION (ELECTRO STATICS)

From Gauss' law

$$\oint_S \bar{D} \cdot d\bar{s} = Q$$

per unit volume

$$\frac{\oint_S \bar{D} \cdot d\bar{s}}{\Delta v} = \frac{Q}{\Delta v}$$

As the volume shrinks to zero

$$\lim_{\Delta v \to 0} \frac{\oint \bar{D} \cdot d\bar{s}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

The above equation, on the left is div $\bar{D}$ and on the right volume charge density

$$\boxed{\text{div } D = \rho_v}$$

(or) $\boxed{\nabla \cdot D = \rho_v}$

divergence of a vector $\bar{A}$ at a given point is the outward flux per unit volume as the volume shrinks about P.

$$\text{div } \bar{A} = \nabla \cdot \bar{A} = \lim_{\Delta v \to 0} \frac{\oint A \cdot ds}{\Delta v}$$

This is the first of Maxwell's four equations as they apply to electro and steady magnetic fields.

Statement! Electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there.

This equation is aptly called the point form of Gauss' law.

Gauss' law relates the flux leaving any closed surface to the charge enclosed, and Maxwell's first equation makes an identical stateme on a per unit volume basis for a vanishingly small volume or a poi

Because that the divergence may be expressed as the sum of three partial derivatives, Maxwell's first equation is also described as Th differential-equation form of Gauss's law, and conversely, Gauss' la is recognized as the integral form of Maxwell's first equation

Energy Expended in Moving a Point Charge in an Electric Field.

(or)

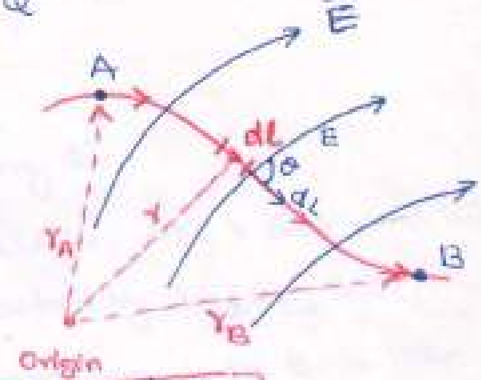
Work done in Moving a point Charge in an Electric Field.

Electric field intensity was defined as the force on a unit test charge at that point at which we wish to find the value of this vector field. If we attempt to move the test charge against the electric field we have to exert a force equal and opposite to that exerted by the field, and this requires us to expand energy or do work.

If we wish to move the charge in the direction of the field, our energy expenditure (work done) turns out to be ~~zero~~negative; we do not do the work, the field does.

Suppose we wish to move a point charge Q from point A to point B in an electric field $\bar{E}$ as shown in figure. From the Coulomb's law the force on Q is

$$\bar{F} = Q\bar{E}.$$



So that the work done in displacing the charge by $d\bar{L}$ is

$$\boxed{dw = -\bar{F}.d\bar{L} = -Q\bar{E}.d\bar{L}}$$

∴ Total work done

$$\boxed{W = -Q\int_A^B \bar{E}.d\bar{L}}$$

-ve sign indicates that the work is done by an external agent.

This differential amount of work required may be zero under several conditions determined from above eq. i.e. $-QEdL\cos\theta$

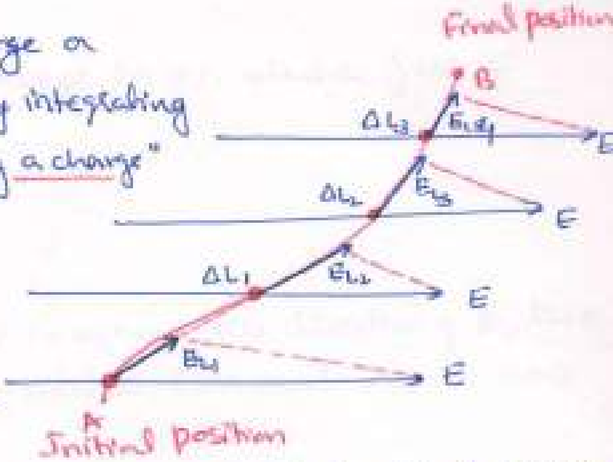
if ~~cosθ~~ = π/2, cosθ = 0 ∴ the energy will be zero.

There are trivial conditions for which E, Q, or $d\bar{L}$ is zero, and a much more important case in which $\bar{E}$ and $d\bar{L}$ are perpendicular as shown above. (θ = π/2) Here the charge is moved always in a direction at right angles to the electric field.

* Anology between Electric Field & Gravitational Field, where again work must be done to move against the field.

* Sliding a mass around with constant velocity on a frictionless surface is an effortless process if the mass is moved along a constant elevation contour; +ve or -ve work must be done in moving it to a higher or lower elevation, respectively.

The work required to move the charge a finite distance must be determined by integrating "work done by electric field in moving a charge"



$$\boxed{W = -Q\int_{int}^{final} \bar{E}\cdot d\bar{L}}$$

where path must be specified before the integral can be evaluated. The charge is assumed to be at rest at both its initial & final position

This definite integral is basic to field theory.

For the analysis the above equation can be rewritten for the path shown in the figure.

$$W = -Q\int_{int}^{final} E_L dL$$

where E_L is the component of $\bar{E}$ along $d\bar{L}$.

The path is divided into 3 segments ΔL_1, ΔL_2 & ΔL_3 and the component along the segment are denoted by E_{L_1}, E_{L_2} & E_{L_3}.

The work involved in moving a charge Q from A to B is then approximately

$$W = -Q(E_{L_1}\Delta L_1 + E_{L_2}\Delta L_2 + E_{L_3}\Delta L_3)$$

or using vector notation

$$W = -Q(\bar{E}_1\cdot\Delta\bar{L}_1 + \bar{E}_2\cdot\Delta\bar{L}_2 + \bar{E}_3\cdot\Delta\bar{L}_3)$$

Since we have assumed a uniform field

$$\bar{E}_1 = \bar{E}_2 = \bar{E}_3$$

$$\boxed{\therefore W = -Q\bar{E}\cdot(\Delta\bar{L}_1 + \Delta\bar{L}_2 + \Delta\bar{L}_3)}$$

$$\boxed{W = -Q\bar{E}\cdot\bar{L}_{AB}} \quad \text{uniform } \bar{E}$$

$$d\bar{L} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$$

$$d\bar{L} = d\rho\,\bar{a}_\rho + \rho\,d\phi\,\bar{a}_\phi + dz\,\bar{a}_z$$

$$d\bar{L} = dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi$$

or $W = -Q\int_A^B \bar{E}\cdot d\bar{L}$ -

as applied to uniform field

$$\boxed{W = -Q\bar{E}\cdot\int_A^B d\bar{L} = -Q\,\bar{E}\cdot\bar{L}_{AB}}$$

where $\bar{L}_{AB} = \int_A^B d\bar{L}$

Note! The work involved in moving the charge depends only on Q, E, & $\bar{L}_{AB}$ a vector drawn from the initial to the final point of the path chose

Note on Work Done

The work done in moving a point charge in an electric field $\bar{E}$ from position A to B is given by

$$W = -Q\int_A^B \bar{E}.d\bar{L}$$

1. When the movement of the charge Q is against the direction of $\bar{E}$, then the work done is $\overset{-ve}{\text{~~positive~~}}$, which indicates external source has the work.

2. When the movement of the charge Q is in the direction of $\bar{E}$, then the work done is $\overset{+ve}{\text{~~negative~~}}$, which indicates field itself has done the work, no external source is required.

3. The work done is independent of the path selected from A to B but depends on end points A & B

4. When the path selected is such that it is always perpendicular to $\bar{E}$ i.e. the force on the charge is always exerted at right angles to the direction in which charge is moving, then the work done is zero. This indicates θ, the angle between $\bar{E}$ & $d\bar{L}$ is 90°. $\therefore\ \bar{E}.d\bar{L} = EdL\cos\theta = 0.\ (\because \theta = 90^\circ)$

5. If the path selected is such that it is forming a closed contour, i.e., starting point is same as the terminating point then the work done is zero. (Kirchoff's Voltage Law)

Electric Potential

Electric potential tells us how large the energy of a small charge at a point of the electric field is, per unit charge.

In the last section, it has been discussed that the work done in moving a point charge Q from A to B in the electric field $\bar{E}$ is given by

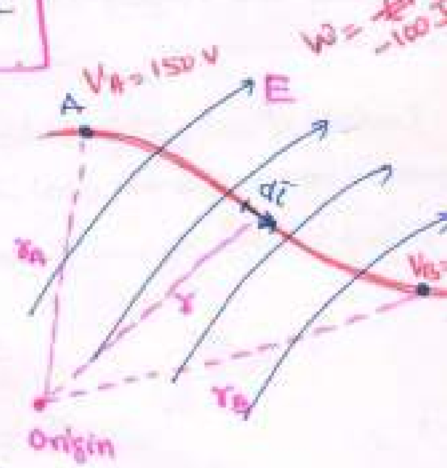
$$W = -Q\int_A^B \bar{E}.d\bar{L} = Q.V_{AB}$$ Q times V_{AB} $W = 100Q$

$$W = Q[V_B - V_A] = QV_B - QV_A$$

If the charge Q is selected as unit test charge then from the ab equation, we get work done in moving unit charge from A to B in th Field $\bar{E}$. This work done in moving the unit charge from point A to B in the field $\bar{E}$ is called potential difference between points A & B It is denoted by V.

$$\boxed{\therefore \text{ Potential Difference } V_{AB} = -\int_A^B \bar{E}.d\bar{L}}$$

Potential Due to a Point Charge.



If the Field $\bar{E}$ is due to the point charge Q located at the origin, then

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 r^2}\bar{a_r}$$

Now consider a unit charge which is placed at a point A is at radial distance r_A from the origin. It is moved against the direction of the electric field direction $\bar{E}$ from point A to B. The point B is at a radial distance r_B from the origin $V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2} a_r . d\bar{L}$

$$\therefore V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2}\bar{a_r}.dr\bar{a_r}$$

in spherical coordinates

where $d\bar{L} = dr\bar{a_r} + rd\theta\bar{a_\theta} + r\sin\theta\, d\phi$

$$= -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2}dr \qquad V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2}\bar{a_r}.(dr\bar{a_r} + rd\theta\bar{a_\theta} + r\sin$$

Since E points in the radial dir any contribution from a displacement θ or φ direction is wiped out by the dot product

$$= \frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_{r_A}^{r_B} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$\boxed{V_{AB} = V_B - V_A}$$

$\therefore V_{AB}$ may be regarded as the potential at B w.r.t A.

$\bar{E}.d\bar{l} = E\cos\alpha\, dl$
$= E dr$
where α is the angle between E & dl

~~total distance~~ (or)

potential difference
or total distance travelle

$$V_{A\infty} - V_{B\infty}$$

$$\int_A^{\infty} \underbrace{E.dl}_{x_A} - \int_B^{\infty} \underbrace{E.dl}_{x_B}$$

$$[x_A]_A^{\infty} - [x_B]_B^{\infty}$$

$$\cancel{x_\infty} - x_A - (\cancel{x_\infty} - x_B)$$

$$x_{BA} = x_B - x_A$$

$$V_{BA} = V_B - V_A$$

In problems involving point charges, it is customary to choose infinity as reference; that is, we assume the potential at infinity is zero. Thus if $V_A = 0$ as $r_A \rightarrow \infty$, the potential at any point $(r_B \rightarrow r)$ due to the point charge Q located at the origin is

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

Hence from the above derivation, the potential difference V_{AB} is independent of the path as asserted earlier as the contribution from displacements in the θ and ϕ directions are wiped out.

In general, vectors whose line integrals does not depend on the path of integration are called conservative. Thus $\bar{E}$ is conservative.

Definition of Potential

The potential at any point is the potential difference between that point and a chosen point (or reference point) at which the potential is zero.

In other words, by assuming zero potential at infinity, the potential at a distance r from the point charge is the work done per unit charge by an external agent in transferring a test charge from infinity to that point. Thus

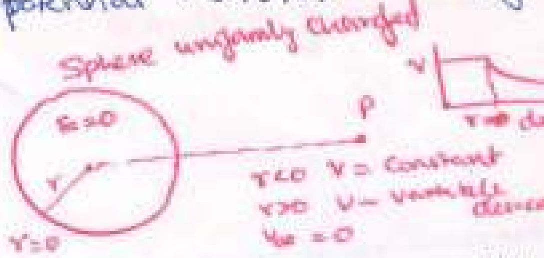
$$V = -\int_{\infty}^{r} \bar{E}\cdot d\bar{l}$$

$$V_A = \int_{A}^{\infty} \bar{E}\cdot d\bar{l} = -\int_{\infty}^{A} \bar{E}\cdot d\bar{l}$$

$$V_B = \int_{B}^{\infty} \bar{E}\cdot d\bar{l} =$$

If the point charge Q is not located at the origin but at a point whose position vector is $\bar{r}'$, the potential $V(x,y,z)$ or simply $V(\bar{r})$ at $\bar{r}$ becomes

$$V(\bar{r}) = \frac{Q}{4\pi\epsilon_0 |\bar{r}-\bar{r}'|}$$



For N point charges $Q_1, Q_2, \dots Q_N$ located at points with positio vectors $\bar{r}_1, \bar{r}_2, \dots \bar{r}_n$, the potentials at $\bar{r}$ is (using superposition principle)

$$V(\bar{r}) = \frac{Q_1}{4\pi\epsilon_0 |\bar{r}-\bar{r}_1|} + \frac{Q_2}{4\pi\epsilon_0 |\bar{r}-\bar{r}_2|} + \dots + \frac{Q_N}{4\pi\epsilon_0 |\bar{r}-\bar{r}_N|}$$

or $$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{N}\frac{Q_k}{|\bar{r}-\bar{r}_k|}$$ (point charges)

For continuous charge distributions

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int_L \frac{\rho_L(\bar{r}')\,dl'}{|\bar{r}-\bar{r}'|} \quad \text{(line charge)}$$

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\rho_s(\bar{r}')\,ds'}{|\bar{r}-\bar{r}'|} \quad \text{(Surface Charge)}$$

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho_v(\bar{r}')\,dv'}{|\bar{r}-\bar{r}'|} \quad \text{(volume Charge)}$$

where the primed coordinates are used customarily to denote
source point location and the unprimed coordinates refer to field point.
(the point at which V is to be determined)

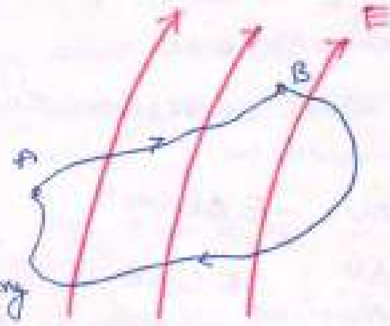
~~Relationship between E & V~~ - Maxwell's Equation (2^{nd} eq)

As we know, the potential difference between points A & B is
independent of the path taken, Hence,

$$V_{BA} = -V_{AB}$$

that is $V_{AB} + V_{BA} = \oint \bar{E}.d\bar{l} = 0$

or $\boxed{\oint_L \bar{E}.d\bar{l} = 0} \longrightarrow ①$



This shows that the line integral of $\bar{E}$ along
a closed path must be zero.
Physically this implies that no work is done in moving a charge
along a closed path in an electrostatic field. Applying Stokes Theore

$$\oint_L \bar{A}.d\bar{l} = \int_S (\nabla\times\bar{A}).d\bar{s}$$

$$\oint_L \bar{E}.d\bar{l} = \int_S (\nabla\times\bar{E}).d\bar{s} = 0$$

(or) $(\nabla\times E)\cos\theta\, ds = 0$

$$\Rightarrow \boxed{(\nabla\times E) = 0} \longrightarrow ②$$

Any vector field that satisfies eq ① & ② is said to be conserval
or irrotational. In other words, vectors whose line integrals doe
not depend on the path of integration are called conservative
vectors. Thus an electrostatic field is a conservative vector.
Equations ① $\overset{\text{or}}{\&}$ ② is referred to as Maxwell's Equation (2^{nd} eq
for electrostatic fields. Eq ① — integral form
Eq ① — Differential form

Relation between E & V

Consider $\overline{E}$ due to a particular charge distribution in space. The electric field $\overline{E}$ and potential V is changing from point to point in space. Consider a vector incremental length $\Delta\overline{L}$ making an angle θ w.r.t the direction of $\overline{E}$ as shown in fig.

To find the incremental potential we use

$$\Delta V = -\overline{E}\cdot\Delta\overline{L}$$

Now $\Delta\overline{L} = \Delta L\,\overline{a}_L$

where $\overline{a}_L$ - is the unit vector in the direction of ΔL.

the above equation can be written as

$$\Delta V = -(E\overline{a}_L)\cdot(\Delta L\overline{a}_L)$$

$$= -E\Delta L\;\overline{a}_L\cdot\overline{a}_L$$

$$\Delta V = -E\Delta L$$

where E is the component of $\overline{E}$ in the direction of $\overline{a}_L$

(or) The above expression can also be expressed as

$$\Delta V = -E\,\Delta L\cos\theta$$

$$\frac{\Delta V}{\Delta L} = -E\cos\theta$$

To find ΔV at a point, take $\lim \Delta L \to 0$

$$\therefore \lim_{\Delta L\to 0}\frac{\Delta V}{\Delta L} = -E\cos\theta$$

But $\lim\limits_{\Delta L\to 0}\frac{\Delta V}{\Delta L} = \frac{dV}{dL}$ = potential gradient

$$\therefore \frac{dV}{dL} = -E\cos\theta.$$

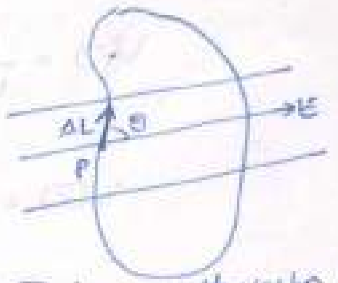
$\frac{dV}{dL}$ can be maximum only when $\cos\theta = -1$ ie $\theta = +180°$. This indicates the ΔL must be in the opposite direction to $\overline{E}$

$$\therefore \left.\frac{dV}{dL}\right|_{max} = E$$

This equation shows:

1). Maximum value of the potential gradient gives the magnitude of the electric field intensity $\overline{E}$

2). The maximum value of rate of change of increment in distance is oppo to the direction of $\overline{E}$.



If $\overline{a}_n$ is a unit vector in the direction of increasing potential then $\overline{E} = -\left.\frac{dV}{dL}\right|_{max}\overline{a}_n$ norma the equip surfa

$\overline{E}$ & potential gradient a in opposite direction.

∴ The maximum value of rate change of potential with dist $\left(\frac{dV}{dL}\right)$ is called gradient of V.

Gradient of V = grad V = ∇V

$$\nabla V = \text{grad}\,V = -\overline{E}\ \text{V/m}.$$

$$\boxed{\overline{E} = -\nabla V = -(\text{grad}\,V)}$$

gradient of a scalar is vector

Potential Gradient   Relationship between $\bar{E}$ & V

We now have two methods of determining potential, one directly from the electric field intensity by means of line integral, and another from the basic charge distribution itself by a volume integral. Neither method is very helpful in determining the fields in most practical problems since neither the electric field intensity nor the charge distribution is very often known. Losses may be calculated if we find the capacitance between the conductors, or charge and current distribution on the conductors from which losses may be calculated.

These quantities may be easily obtained from the potential field, and our immediate goal will be a simple method of finding electric field intensity from the potential.

Already known, general line-integral relationship between these quantities $V = -\int \bar{E}\cdot d\bar{L} \rightarrow$ ①

But this is much easier to use in the reverse direction; given E find V

However eq ① may be applied to a very short element of length $\Delta \bar{L}$ along which $\bar{E}$ is essentially constant, leading to an incremental difference ΔV,

$$dv = -\bar{E}\cdot d\bar{L} = -E_x dx - E_y dy - E_z dz$$

The total change in $V(x,y,z)$ is the sum of the partial changes w.r.t x, y, z variables.

$$\therefore dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz$$

Comparing two expressions for dv, we obtain

$$E_x = -\frac{\partial v}{\partial x}; E_y = -\frac{\partial v}{\partial y}; E_z = -\frac{\partial v}{\partial z}$$

Substitut

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz$$

$$= \left(\frac{\partial v}{\partial x}\bar{a}_x + \frac{\partial v}{\partial y}\bar{a}_y + \frac{\partial v}{\partial z}\bar{a}_z\right)\cdot (dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z)$$

$$= \nabla V\cdot d\bar{L}$$

$$d\bar{L} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$$

(or) $G\cdot d\bar{L}$

∇V

Potential Gradient   Relationship between $\overline{E}$ & V

We now have two methods of determining potential, one directly from the electric field intensity by means of line integral, and another from the basic charge distribution itself by a volume integral. Neither method is very helpful in determining the fields in most practical problems Since neither the electric field intensity nor the charge distribution is very often known. Losses may be calculated if we find the capacitance between the conductors, or charge and current distribution on the conductors from which losses may be calculated.

These quantities may be easily obtained from the potential field, and our immediate goal will be a simple method of finding electric field intensity from the potential.

Already known, general line-integral relationship between these quantities $V = -\int \overline{E}.d\overline{L} \rightarrow ①$

But this is much easier to use in the reverse direction; given E find

However eq ① may be applied to a very short element of length $\Delta \overline{L}$ along which $\overline{E}$ is essentially constant, leading to an incremental difference ΔV,

Substitute

$$dV = -\overline{E}.d\overline{L} = -E_x dx - E_y dy - E_z dz$$

The total change in $V(x,y,z)$ is the sum of the partial changes w.r.t x, y, z variables.

$$\therefore dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$= \left(\frac{\partial V}{\partial x}\overline{a}_x + \frac{\partial V}{\partial y}\overline{a}_y + \frac{\partial V}{\partial z}\overline{a}_z\right) \cdot (dx\,\overline{a}_x + dy\,\overline{a}_y + dz\,\overline{a}_z)$$

$$= \nabla V . d\overline{L}$$

$$d\overline{L} = dx\,\overline{a}_x + dy\,\overline{a}_y + dz\,\overline{a}_z$$

(or) $G.d\overline{L}$

$G = \nabla V$

Comparing two expressions for dV, we obtain

$$E_x = -\frac{\partial V}{\partial x} ; E_y = -\frac{\partial V}{\partial y} ; E_z = -\frac{\partial V}{\partial z}$$

Thus $\boxed{E = -\nabla V}$

$\therefore$ Electric field intensity is the gradient of V. −ve sign shows the direction E is opposite to the direction in which V increases.

Find $\overline{E}$ at the point $P(0,1,1)$ if (a) $V = E_0 e^{-x} \sin\left(\frac{\pi y}{4}\right)$ in Cartesia
(b) $V = E_0 r \cos\theta$ in spherical.

<u>Sol:-</u> (a) $V = E_0 e^{-x} \sin\left(\frac{\pi y}{4}\right)$

$$\overline{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\overline{a_x} + \frac{\partial V}{\partial y}\overline{a_y} + \frac{\partial V}{\partial z}\overline{a_z}\right]$$

$$\frac{\partial V}{\partial x} = E_0 \sin\left(\frac{\pi y}{4}\right)(-1)e^{-x} = -E_0 e^{-x} \sin\left(\frac{\pi y}{4}\right)$$

$$\frac{\partial V}{\partial y} = E_0 e^{-x} \cos\left(\frac{\pi y}{4}\right)\frac{\pi}{4}$$

$$\frac{\partial V}{\partial z} = 0$$

$$\therefore \overline{E} = -\left[-E_0 e^{-x} \sin\frac{\pi y}{4}\ \overline{a_x} + E_0 e^{-x} \cancel{\cos}\frac{\pi}{4}\cos\left(\frac{\pi y}{4}\right)\overline{a_y}\right] V/m$$

at $P(0,1,1)$, $E = E_0\left[0.7071\overline{a_x} - 0.555\overline{a_y}\right]$ V/m.

(b) $V = E_0 r \cos\theta$

$$\overline{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\overline{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\overline{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\overline{a_\phi}\right]$$

$$\frac{\partial V}{\partial r} = E_0\cos\theta,\ \frac{\partial V}{\partial \theta} = -E_0 r\sin\theta,\ \frac{\partial V}{\partial \phi} = 0$$

$$\therefore \overline{E} = -E_0\cos\theta\,\overline{a_r} + E_0\sin\theta\,\overline{a_\theta}\ V/m.$$

Convert $P(0,1,1)$ to spherical coordinates

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{2} \quad \phi = \tan^{-1}\left(\frac{y}{x}\right) = \pi/2 \quad \theta = \cos^{-1}\frac{z}{r} = 45^\circ$$

$$\therefore \overline{E} = E_0\left[-0.7071\,\overline{a_r} + 0.7071\,\overline{a_\theta}\right]\ V/m$$

An electric field is given by $\bar{E} = 6y^2z\,\bar{a}_x + 12xyz\,\bar{a}_y + 6xy^2\,\bar{a}_z$ V/
and $\Delta\bar{l} = -3\bar{a}_x + 5\bar{a}_y - 2\bar{a}_z$ μm. Find the work done in moving a 2
charge along this path if the location of the path is at (a) $P_1(0,3,5)$
at $P_2(1,1,0)$
$P_3(-0.7,-2,0.4)$

Sol: $dw = -Q\bar{E}\cdot\bar{\delta l}$

$= -Q\left[6y^2z\,\bar{a}_x + 12xyz\,\bar{a}_y + 6xy^2\,\bar{a}_z\right]\cdot\left[-3\bar{a}_x + 5\bar{a}_y - 2\bar{a}_z\right]$

$= -Q\{-18y^2z + 60xyz - 12xy^2\}\times 10^{-6}$ δl = μm

$= -2\times10^{-6}\{-18y^2z + 60xyz - 12xy^2\}\times10^{-6}$

(a) at $P_1(0,3,5)$ $dw = -2\times10^{-12}(-180) = 1620$ pJ

(b) at $P_2(1,1,0)$ $dw = -2\times10^{-12}(0+0-12) = 24$ pJ

(c) at $P_3(-0.7,-2,0.4)$ $dw = -2\times10^{-12}(-28.8+33.6+22.6) = -78.6$ pJ

Find the electric field at a point (1,-2,1) m if the given potential $V = 3x^2y + 2yz^2 + 2xyz$.

Sol: $\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}a_z\right]$

$\frac{\partial V}{\partial x} = 6xy + 0 + 2yz$, $\frac{\partial V}{\partial y} = 3x^2 + 2z^2 + 2xz$, $\frac{\partial V}{\partial z} = 0 + 4y + 2xy$

at point (1,-2,1)m $\frac{\partial V}{\partial x} = -16$; $\frac{\partial V}{\partial y} = 7$; $\frac{\partial V}{\partial z} = -12$

$\therefore \bar{E} = -\left[-16\bar{a}_x + 7\bar{a}_y - 12\bar{a}_z\right] = 16\bar{a}_x - 7\bar{a}_y + 12\bar{a}_z$ V/m.

If $V = 2x^2y + 20z - \frac{4}{x^2+y^2}$ V find E, D, ρ_v at P(6,-2.5,3)

Sol:- $E = -\nabla V = -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$

$\frac{\partial V}{\partial x} = 2y(2x) + 0 - 4\left[\frac{-(2x)}{(x^2+y^2)^2}\right]$

$= 4xy + \frac{8x}{(x^2+y^2)^2}$

$\frac{\partial V}{\partial y} = 2x^2 + \frac{8y}{(x^2+y^2)^2}$; $\frac{\partial V}{\partial z} = 0+20-0 = 20$

$\therefore \bar{E}$ at p(6,-2.5,3)

$\bar{E}_p = -\{59.97\bar{a}_x - 71.98\bar{a}_y - 20\bar{a}_z$ V/m.

$\bar{D} = \bar{E}_p\times\epsilon_0 = 0.531\bar{a}_x - 0.637\bar{a}_y - 0.177\bar{a}_z$ nC/m^2

now $\rho_v = \nabla\cdot\bar{D}$; $\bar{D} = \epsilon_0\bar{E}$

$\therefore \rho_v = (\nabla\cdot E)\epsilon_0$

$\nabla\cdot\bar{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$

$= -\left[4y + \frac{(x^2+y^2)^2 8 - 8x\,2(x^2+y^2)(2x)}{(x^2+y^2)^4}\right] - \left[0 + \frac{(x^2+y^2)^2 8 - 8y\,2(x^2+y^2)(2y)}{(x^2+y^2)^4}\right] + 0$

$\nabla\cdot E = -4y - \frac{8}{(x^2+y^2)^2} + \frac{32x^2}{(x^2+y^2)^3} - \frac{8}{(x^2+y^2)^2} + \frac{32y^2}{(x^2+y^2)^3}$

At p(6,-2.5,3)

$\nabla\cdot E = 10.00895$

$\therefore \rho_v = (\nabla\cdot E)\epsilon_0 = 10.00895\times\epsilon$

$= 88.6193$ pC/m^2

Given the potential $V = \frac{10}{r^2} \sin\theta \cos\phi$,

(a) Find the electric flux density D at $(2, \pi/2, 0)$

(b) calculate the work done in moving a $10\mu C$ charge from point A(1, 30°, 12

to B(4, 90°, 60°)

(a) $\overline{D} = \epsilon_0 E$

But
$$E = -\nabla V = -\left[\frac{\partial V}{\partial r}\overline{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\overline{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\overline{a}_\phi\right]$$
$$= \frac{20}{r^3}\sin\theta\cos\phi\,\overline{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\,\overline{a}_\theta + \frac{10}{r^3}\sin\phi\,\overline{a}_\phi$$

at $(2, \pi/2, 0)$
$$\overline{D} = \epsilon_0\overline{E}(r=2, \theta=\pi/2, \phi=0) = \epsilon_0\left(\frac{20}{8}\overline{a}_r - 0\overline{a}_\theta + 0\overline{a}_\phi\right)$$
$$= 2.5\epsilon_0\overline{a}_r \text{ C/m}^2 = 22.1\,\overline{a}_r \text{ pC/m}^2$$

(b) Since V is known, this method is much easier
$$W = -Q\int_A^B E \cdot dL = Q V_{AB} = Q(V_B - V_A)$$
$$= 10\left[\frac{10}{16}\sin 90^\circ \cos 60^\circ - \frac{10}{1}\sin 30^\circ \cos 120^\circ\right] \times 10^{-6}$$
$$= 10\left(\frac{10}{32} - \frac{-5}{2}\right) \times 10^{-6} = 28.125\,\mu J$$

Two point charges $-4\mu C$ and $5\mu C$ are located at (2, -1, 3) and (0, 4, -2 respectively. Find the potential at (1, 0, 1) assuming zero potential at infinity.

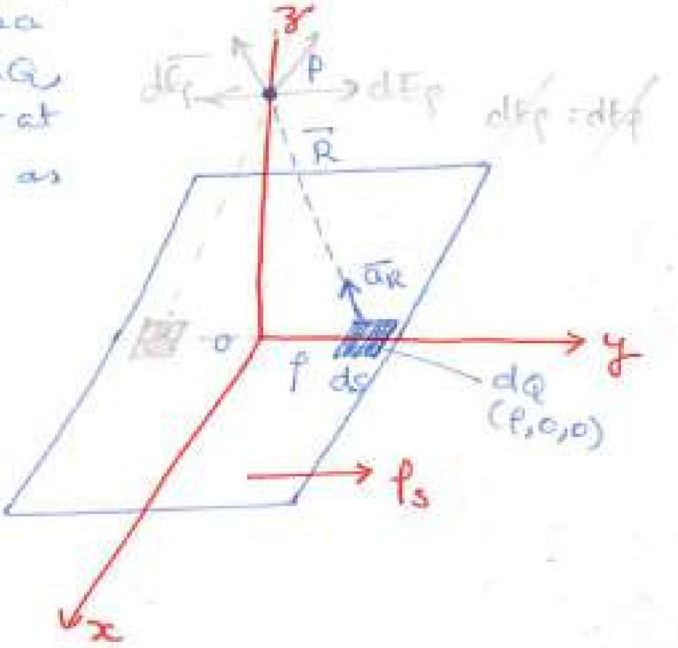
let $Q_1 = -4\mu C$, $Q_2 = 5\mu C$.
$$V(r) = \frac{Q_1}{4\pi\epsilon_0|\overline{r} - \overline{r}_1|} + \frac{Q_2}{4\pi\epsilon_0|\overline{r} - \overline{r}_2|} + C_0$$

If $V(\infty) = 0$, $C_0 = 0$
$$|\overline{r} - \overline{r}_1| = |(1,0,1) - (2,-1,3)| = |(-1,1,-2)| = \sqrt{6}$$
$$|\overline{r} - \overline{r}_2| = |(1,0,1) - (0,4,-2)| = |(1,-4,3)| = \sqrt{26}$$

Hence
$$V(1,0,1) = \frac{10^{-6}}{4\pi \times \frac{10^{-4}}{36\pi}}\left[\frac{-4}{\sqrt{6}} + \frac{5}{\sqrt{26}}\right]$$
$$= 9 \times 10^{3}(-1.633 + 0.9806)$$
$$= -5.872 \text{ kV}$$

EFI due to an Infinite Sheet of Charge.

Consider an infinite sheet of charge placed in the x-y plane with uniform charge density ρ_s C/m². Let the differential charge on a differential surface area ds on the x-y plane be dQ, and let point P be a point at a distance z on the z-axis as shown in figure.



Using cylindrical coordinates, the position vector of dQ has coordinates $(\rho,0,0)$, and that of P is $(0,0,z)$

The differential charge is

$$dQ = \rho_s ds = \rho_s \rho d\rho d\phi.$$

The distance vector is

$$\vec{R} = -\rho\bar{a}_\rho + z\bar{a}_z$$

$$\vec{R} = \sqrt{R^2+z^2}$$

and $\bar{a}_R = \dfrac{-\rho\bar{a}_\rho + z\bar{a}_z}{\sqrt{\rho^2+z^2}}$

∴ differential EFI is

$$d\bar{E} = \frac{\rho_s ds}{4\pi\epsilon_0 R^2}\bar{a}_R$$

$$= \frac{\rho_s \rho d\rho d\phi}{4\pi\epsilon_0(\rho^2+z^2)^{3/2}}(-\rho\bar{a}_\rho + z\bar{a}_z)$$

Since the sheet is symmetrical about the radial distance, $\bar{a}_\rho$ components will be canceled,

$$\therefore d\bar{E} = \frac{\rho_s \rho d\rho d\phi}{4\pi\epsilon(\rho^2+z^2)^{3/2}}(z\bar{a}_z)$$

The total EFI

$$\bar{E} = \int_s d\bar{E} = \int_0^{2\pi}\int_0^{\infty}\frac{\rho_s \rho d\rho d\phi}{4\pi\epsilon_0(\rho^2+z^2)^{3/2}} z\bar{a}_z$$

For an infinite sheet charge, the limits for ρ are from 0 to ∞ and for ϕ, they are from 0 to 2π.

$$\therefore \bar{E} = \frac{\rho_s z}{4\pi\epsilon_0}\int_0^{\infty}\int_0^{2\pi}\frac{\rho d\rho d\phi\ \bar{a}_z}{(\rho^2+z^2)^{3/2}}$$

$$= \frac{\rho_s z}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\rho d\rho}{(\rho^2+z^2)^{3/2}}\left[\phi\right]_0^{2\pi}\bar{a}_z$$

$$= \frac{\rho_s z}{2\epsilon_0}\int_0^{2\pi}\frac{\rho d\rho}{(\rho^2+z^2)^{3/2}}$$

Let $\rho^2+z^2 = t^2$, $2\rho d\rho = 2t dt$

limits are from z to ∞

$$\therefore \bar{E} = \frac{\rho_s z\bar{a}_z}{2\epsilon_0}\int_z^{\infty}\frac{t dt}{t^3}$$

$$= \frac{\rho_s z\bar{a}_z}{2\epsilon_0}\int_z^{\infty}\frac{1}{t^2}dt$$

$$= \frac{\rho_s z\bar{a}_z}{2\epsilon_0}\left(\frac{1}{z}\right)$$

$$\boxed{\therefore \bar{E} = \frac{\rho_s}{2\epsilon_0}\bar{a}_z \ \text{V/m}}$$

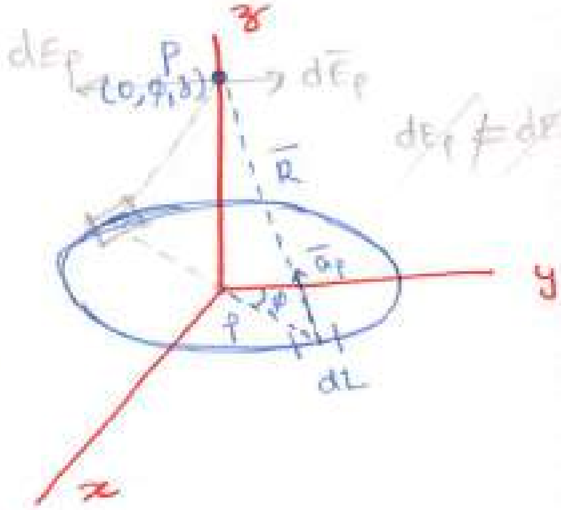
Note 1: EFI has component normal to the plane of sheet of charge.

2. If the point P is considered on the negative axis, then the field strength becomes $\bar{E} = \dfrac{-\rho_s}{2\epsilon_0}\bar{a}_z$ V/m.

∴ EFI is always normal to the sheet $\boxed{\bar{E} = \dfrac{\rho_s}{2\epsilon_0}a_n \ \text{V/m}}$

EFI due to a Circular Ring of Charge.

Consider a circular ring of radius ρ placed in the x-y plane with its centre at the origin, carrying charge Q. Let the charge density be ρ_ℓ c/m. As per the symmetry, cylindrica coordinate system is chosen.



Let a point $P(0, \phi, z)$ be on the z-axis, perpendicular to the ring plane as shown in Figure.

Let dQ be the differential charge on a differential length dl on the circular path. The differential charge is $dQ = \rho_L dL$.

The position vector of dQ has coordinates $(\rho, \phi, 0)$

From the figure $dL = \rho\, d\phi$

The distance vector $\bar{R} = (0-\rho)\bar{a}_\rho + (z-0)\bar{a}_z$

$$= -\rho\bar{a}_\rho + z\bar{a}_z$$

and $|\bar{R}| = \sqrt{\rho^2+z^2}$

The unit vector $\bar{a}_R = \dfrac{-\rho\bar{a}_\rho + z\bar{a}_z}{\sqrt{\rho^2+z^2}}$

$\therefore$ The differential Field strength $d\bar{E} = \dfrac{\rho_L\, dL}{4\pi\epsilon R^2}\bar{a}_R$

$$d\bar{E} = \frac{\rho_L \rho\, d\phi}{4\pi\epsilon_0(\rho^2+z^2)} \quad \frac{-\rho\bar{a}_\rho + z\bar{a}_z}{\sqrt{\rho^2+z^2}}$$

$$= \frac{-\rho_L \rho^2 d\phi\, \bar{a}_\rho}{4\pi\epsilon_0(\rho^2+z^2)^{3/2}} + \frac{\rho_L \rho z\, d\phi}{4\pi\epsilon_0(\rho^2+z^2)}$$

Since the radial component of $\bar{E}$ at point P are symmetrica about the x-y plane and are cancelled out, the radial component of $\bar{a}_\rho$ is zero.

$$\therefore d\bar{E} = \frac{\rho_L \rho z\bar{a}_z}{4\pi\epsilon_0(\rho^2+z^2)^{3/2}} d\phi$$

$\therefore$ Total EFI is

$$\bar{E} = \int_0^{2\pi} \frac{\rho_L \rho z\bar{a}_z}{4\pi\epsilon_0(\rho^2+z^2)^{3/2}} d\phi = \frac{\rho_L \rho z}{2\epsilon_0(\rho^2+z^2)^{3/2}}\bar{a}_z \ \text{V/m}.$$

Note! If $z = 0$, the EFI $\bar{E} = 0$. That is, if a point charge is kep at the centre of the ring on the same plane, the force on the point charge is zero.

Assignment

1. A point charge at $Q_1 = 600\mu c$ located at $(2,-4,-3)$ experience force $\overline{F} = 4\overline{a_x} - 4\overline{a_y} + 2\overline{a_z}$ due to a point charge Q_2 at $(4,-6,-4)$. Determine the magnitude of Q_2.

2. Two point charges, $Q_1 = 35\mu c$ and $Q_2 = 60\mu c$, are located at $(-4,6,-8)$ and $(3,5,2)$ respectively. Find force on Q_1

3. Four point charges are located in free space as shown in figure (1). Find force experienced by the $1\mu c$ charge.

4.

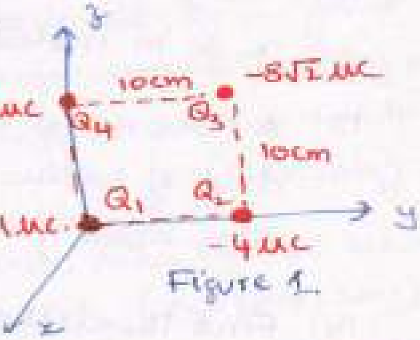


Figure 1.

4. Four point charges of $500\mu c$ each are placed at the corners of a square of $3\sqrt{2}$ m side. The square is located in the $z=0$ plane between $x = \pm 3/\sqrt{2}$ and $y = \pm 3/\sqrt{2}$ m in free space. Find the force on a point charge $30\mu c$ located at $(0,0,4)$ m (Feb 2008)

5. It is required to hold four equal point charges in equilibri at the corners of a square. Find the point charge which wi do this if placed at the centre of the square. (Nov 2008, May 2011)

6. Two identical conducting spheres have charge of 2nc and $-0.5nc$ respectively. When they are placed 4cm apart, what is the force between them? If they are brought into conta and then separated by 4 cm, what is the force between them? (Nov 2005)

7. Concentrated charges of $0.25\mu c$ are located at the vertices of an equilateral triangle of 10 m side. Find the magnitude and direction of the force on one charge due to the other two charges. (Nov 2010)

8. Derive an expression for electric field intensity at any point p in the x-y plane at a radial height 'h' from a finite line charge of λ c/m extending along the z-axis from 32 to 33 m. (Nov 2008)

9. Calculate the work done in moving a point charge of 10 μC from point (4, 90°, 60°) to (3, 30°, 120°) if $V = \frac{10}{r} \cos\theta \sin\phi$ volts.

10. Six equal point charges, Q = 100nC, are located at x = 5, 6, 7, 8, 9 and 10 m. Find the potential at the origin.

11. A potential function is given by $V = 4/(x^2+y^2+z^2)$. Find the expression for $\bar{E}$. (Nov 05)

12. The potential for a certain region is given by $V = \frac{300}{x}$ V. where x is in meters. Find the electric field at the point P (x = 1m) (Feb 08, Nov 07)

13. A 300nC of charge is uniformly distributed around a circular disc of radius 4m. Find the potential at a point on the axis 5m apart from the plane of the ring.

14. Find the electric field at any point between two concentric spherical shells, the inner spherical shell having a charge Q_1 and the outer charge Q_2 (Nov 2010, May 2011.)

15. Given that $D = 500\, e^{-0.1x}\, \bar{a}_x$ μC/m², find the electric flux crossing the surfaces of area 1m² normal to the x-axis and located at x = 1m and x = 5m (March 2001)

16. The flux density $\bar{D} = \frac{r}{3} \bar{a}_r$ nc/m² in free space. Find
    (i) $\bar{E}$ at r = 0.2m
    (ii) The total electric flux leaving the sphere of r = 0.2m (May 2011)
    (iii) The total charge within the sphere of r = 0.3m.

⑮ $\psi = \int_s D.ds$ $D = 500\, e^{-0.1x}\, \bar{a}_x \times 10^{-6}$ c/m²

$ds = dy\,dz\,\bar{a}_x$ normal to yz plance

Assuming that center of surfaces are on the x-axis

$\psi = Ds$

at x = 1m $\psi = 500\, e^{-0.1} \times 10^{-6} = 452.2$ μC

at x = 5m $\psi = 500\, e^{-0.5} \times 10^{-6} = 301.2$ μC.

1. State and explain Coulomb's law of electrostatic fields in vector form. (May 2008, 2012)

2. State ~~Colom~~ Coulomb's law of force between any two point charges and states the units of quantities (May 2008, 2010, 2011)

3. Derive the concept of electric field intensity from Coulomb's law (Nov 2008, ~~2010~~, May 2012

4. Obtain an expression for the total force experienced by a point charge due to infinite number of point charges around it (May 201

5. Obtain an expression for electric field intensity at a point due to infinite number of point charges (Nov 2010)

6. Derive an expression for $\bar{E}$ due to an infinite line charge alon the z-axis at an arbitrary point P (x,y,z). (Nov 2008, 2011)

7. A line charge density is uniformly distributed over a length 2a with centre as origin along the x-axis. Find E at a point P on the z-axis at a distance d. (Nov 2010)

8. Find the electric field at any point between the two concentric spherical shells, the inner spherical shell having a charge Q_1 and outer spherical shell a charge Q_2 (May 2012)

9. State and explain Gauss's law. (Nov 2009, 2010)

10. Derive the Maxwell's first equation as applied to the electrostatic using Gauss' law (May 2010, 2012)

11. Derive the expression for D due to an infinite sheet of charge ~~using Gauss's law~~. placed in z=0 plane using Gauss law (May 2010)

12. Explain the Gauss's law applied to the case of infinite line charge and derive the expression for D due to the infinite line charge. (May 2007, 2011)

~~13. It is required to hold~~

13. Prove that electric field intensity is equal to the negative gradi of the potential i.e. $E = -\nabla V$, where V is the potential. (May 2007, 2011)

14. Define ~~and~~ explain the following terms.
    (a) electric flux density (D)
    (b) Electric field intensity (E)
    (c) Electrostatic Energy
    (E) electric potential (May 201

15. Derive Gauss' law from Coulomb's law (Nov 2011)

Unit II: Dielectrics and Capacitance

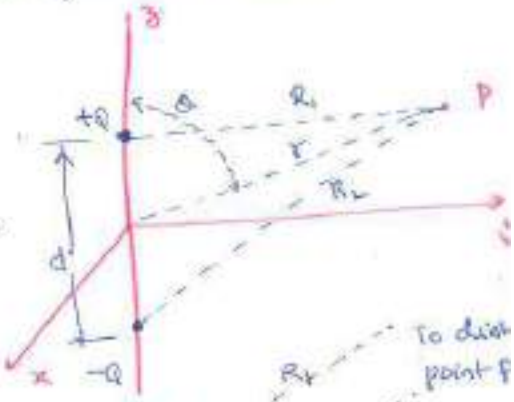
- **Electric Dipole, Dipole Moment, Torque**
- **Classification of Materials**
  - Conductors
  - Semi-conductors
  - Insulators or Dielectrics
- **Electric Field in Material Space (Conductors & Dielectrics)**
- **Dielectric Materials**
  - Polarization
  - Dipole Moment
  - Properties of Dielectrics
  - Dielectric Constant
  - Dielectric Strength
  - **D** in Dielectrics
- **Boundary Conditions**
  - Between Conductor – Free Space (Conductor)
  - Between Conductor – Dielectric
  - Between Dielectric – Dielectric
  - Relationship between **E** & **D** at the boundary
- **Capacitance**
  - Parallel Plate
  - Coaxial Cable/Capacitor
  - Coaxial Cable with Two Dielectrics
  - Spherical Capacitor
  - Composite Parallel Plate Capacitor
    - Boundary Parallel to the Plate
    - Boundary Normal to the Plate
- **Energy Density in the Electrostatic Field**
  - Energy Stored in a Capacitor
    - Parallel Plate Capacitor
    - Co-axial Capacitor
- **Current and Current Density**
  - Types of Currents
  - Point form of Ohm's Law
  - Continuity Equation of Current
  - Relaxation Time
- **Resistance of a Conductor**

THE ELECTRIC DIPOLE

An electric dipole, or simply a dipole, is the name given to a two point charges of equal magnitude and opposite sign, separated by a distance which is small compared to the distance to the point P at which we want to know the electric and potential fields.

An electric dipole is formed when two point charges of equal magnitude but opposite sign are separated by a small distance.

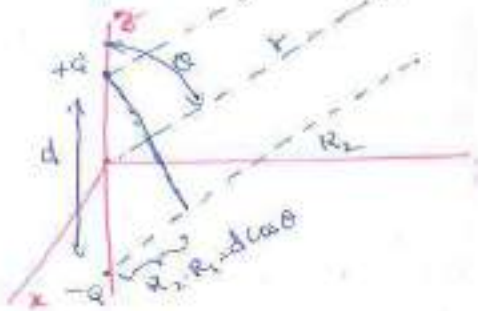
The dipole fields which we shall develop in this section are quite important because they form the basis for the behaviour of dielectric materials in electric fields.



From the figure, let the distances from Q and −Q to P be R_1 and R_2 respectively, then the total potential is

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) = \frac{Q}{4\pi\epsilon_0}\,\frac{R_2-R_1}{R_1R_2}$$

Note that the plane $z=0$, midway between the two point charges, is the locus of points for which $R_1=R_2$ and is therefore at zero potential, as are all points at infinity.



For a distant point $R_1=R_2$ and the product $R_1R_2 = r^2$. But in numerator R_2-R_1 may be approximated if R_1 and R_2 are assumed to be parallel (from fig (b)) Otherwise $V=0$

$\therefore R_2-R_1 = d\cos\theta$

Then $\boxed{V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}}$ potential at P due to dipole

The plane $z=0$ $(\theta=90°)$ is at zero potential

Using the gradient relationship in spherical coordinates,

$$\overline{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\overline{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\overline{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\overline{a}_\phi\right)$$

we obtain $\overline{E} = -\left(-\frac{Qd\cos\theta}{2\pi\epsilon_0 r^3}\overline{a}_r - \frac{Qd\sin\theta}{4\pi\epsilon_0 r^3}\overline{a}_\theta\right)$

$V = \frac{1}{4\pi\epsilon_0|\overline{r}-\overline{r}'|^2}\,\overline{P}\cdot\frac{(\overline{r}-\overline{r}')}{|\overline{r}-\overline{r}'|}$ where $\overline{r}$ locates the field point P and $\overline{r}'$ determines the dipole center.

or $\boxed{E = \frac{Qd}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\overline{a}_r + \sin\theta\,\overline{a}_\theta\right)}$ Electric field produced at P due to dipole

Since $d\cos\theta = \overline{d}\cdot\overline{a}_r$ where $\overline{d} = d\overline{a}_z$ then the product $Q\overline{d}$ is called dipole moment and denoted as $\overline{p} = Q\overline{d}$

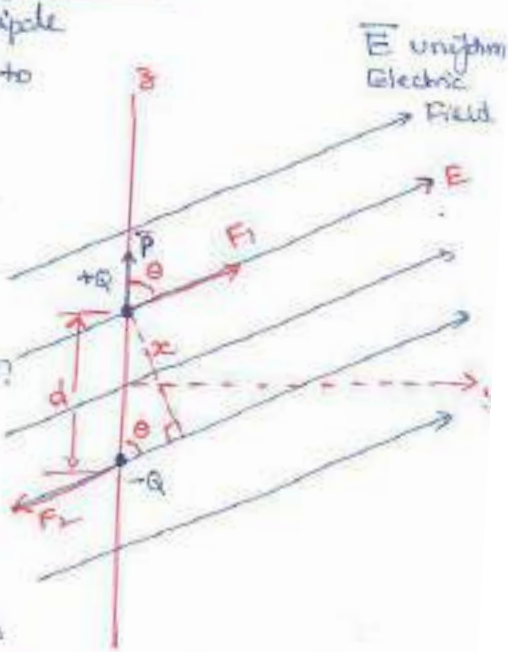
$\boxed{\therefore V = \frac{\overline{p}\cdot\overline{a}_r}{4\pi\epsilon_0 r^2}\,V}$ & $\boxed{\overline{E} = \frac{|\overline{p}|}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\overline{a}_r + \sin\theta\,\overline{a}_\theta\right]}$

Torque Experienced by a Dipole in a Uniform Field

We have seen earlier that $\bar{p} = Q.\bar{d}$, the dipole moment is a vector directed from negative to positive charge forming a dipole.

We will consider the situation of a dipole in a uniform field $\bar{E}$ at an angle θ with the dipole axis.

What happens when a dipole is placed in a uniform field? Will it experience a force?



The two charges in the dipole +Q & -Q experiences a force equal in magnitude to QE but oppositely directed, with the result that the dipole experiences no translation, as the forces F1 & F2 neutralizes each other these forces experience a torque

Torque = Force × Perpendicular distance of separation. F × x

$$T = QE\,x$$

$$= QE\,d\sin\theta$$

$$= Qd\,(E\sin\theta)$$

$$= pE\sin\theta$$

x – perpendicular distance of separation between the two forces.

= d sinθ

can be represented in vector form

$$\boxed{\bar{T} = \bar{p} \times \bar{E}}$$

$$|\bar{T}| = |p|E\sin\theta$$

Example

Two dipoles with dipole moments $-5\bar{a}_z$ nC/m and $9\bar{a}_z$ nC/m are located at points (0,0,-2) and (0,0,3) respectively. Find the potential at the origin.

$$V = \sum_{k=1}^{2} \frac{\bar{P}_k \cdot \bar{r}_k}{4\pi\epsilon_0 r_k^3} = \frac{1}{4\pi\epsilon_0}\left[\frac{\bar{P}_1 \cdot \bar{r}_1}{r_1^3} + \frac{\bar{P}_2 \cdot \bar{r}_2}{r_2^3}\right]$$

where

$\bar{P}_1 = -5\bar{a}_z$ $\quad r_1 = (0,0,0) - (0,0,-2) = 2\bar{a}_z$ $\quad r_1 = |\bar{r}_1| = 2$

$\bar{P}_2 = 9\bar{a}_z$ $\quad r_2 = (0,0,0) - (0,0,3) = -3\bar{a}_z$ $\quad r_2 = |\bar{r}_2| = 3$

Hence

$$V = \frac{1}{4\pi \frac{10^{-9}}{36\pi}}\left[\frac{-10}{2^2} - \frac{27}{3^3}\right] 10^{-9}$$

$$= \underline{-20.25\,V}$$

Prob: A dipole having a moment $\bar{p} = 10\bar{a}_x - 3\bar{a}_y + 5\bar{a}_z$ nC.m is located at $Q(2,4,1)$ in free space. (a) Find V at $P(x,y,z)$. (b) Find V at $(5,1,0)$

Sol:

(a) potential due to dipole $V = \frac{1}{4\pi\epsilon_0 |\bar{r}-\bar{r}'|^2}\, \bar{p} \cdot \frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|}$ $\quad V = \frac{1}{4\pi\epsilon_0 R^3}\, \bar{p}\cdot\bar{a}$

Here $\bar{r}-\bar{r}' = (x\bar{a}_x + y\bar{a}_y + z\bar{a}_z) - (2\bar{a}_x + 4\bar{a}_y + \bar{a}_z) = \bar{R}$

$= (x-2)\bar{a}_x + (y-4)\bar{a}_y + (z-1)\bar{a}_z$

The magnitude is

$$|\bar{r}-\bar{r}'| = \sqrt{(x-2)^2 + (y-4)^2 + (z-1)^2} = |\bar{R}|$$

The potential at $P(x,y,z)$ is

✓ $V(x,y,z) = \frac{\bar{p}\cdot(\bar{r}-\bar{r}')}{4\pi\epsilon_0 |\bar{r}-\bar{r}'|^3}$

$$= \frac{\{10\bar{a}_x - 3\bar{a}_y + 5\bar{a}_z\} \cdot \{(x-2)\bar{a}_x + (y-4)\bar{a}_y + (z-1)\bar{a}_z\} \times 10^{-9}}{4\pi \times 8.854\times10^{-12}\{(x-2)^2 + (y-4)^2 + (z-1)^2\}^{3/2}}$$

$$= 8.996\, \frac{10x - 3y + 5z - 13}{\{(x-2)^2 + (y-4)^2 + (z-1)^2\}^{3/2}}\ V$$

(b) The potential at $(5,1,0)$

$$V(5,1,0) = 8.976 \times \frac{10\times5 - 3\times1 + 5\times0 - 13}{\{(5-2)^2 + (1-4)^2 + (0-1)^2\}^{3/2}} = \underline{2.69V}$$

Prob: point charges 1μC and −1μC are located at (0,0,0.5) and (0,0,−0.5) respectively. Treating these two charges as the dipole at the origin. Calculate: (a) V at P(3,0,4); (b) |E| at P. Now find the exact values for (c) V at P and (d) |E| at P.

Sol: Given point charges of 1μC and −1μC at (0,0,0.5) and (0,0,−0.5) respectively.

(a) Treating these charges as dipole at the origin, the potential at point P(3,0,4) is

$d = 1.0$, $Q = 1.0\times10^{-6}$

$\cos\theta = \frac{4}{3^2+4^2} =$

$$V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{1\times10^{-6}\times1.0}{4\pi\epsilon_0(3^2+4^2)} \times \frac{4}{\sqrt{3^2+4^2}}$$

$= 287.58V.$

(b) The electric field intensity at point P is

$$\bar{E} = \frac{Qd}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\bar{a}_r + \sin\theta\,\bar{a}_\theta\right]$$

$$= \frac{10^{-6}\times1.0}{4\pi\times\epsilon_0\times(3^2+4^2)^{3/2}}\left[2\times\frac{4}{5}\bar{a}_r + \frac{3}{5}\bar{a}_\theta\right]$$

$= 115.152\,\bar{a}_r + 43.184\,\bar{a}_\theta$ V/m

$\therefore |E| = \sqrt{115.152^2 + 43.184^2} = 122.98$ V/m

(c) Here R_1 is the distance between the point (0,0,0.5) and P(3,0,4)

$$\therefore R_1 = |\bar{R}_1| = \sqrt{(3-0)^2+(4-0.5)^2} = 4.6098$$

R_2 is the distance between the point (0,0,−0.5) and P(3,0,4)

$$\therefore R_2 = |\bar{R}_2| = \sqrt{(3-0)^2+(4+0.5)^2} = 5.408$$

Thus the exact value of potential at P(3,0,4)

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{10^{-6}}{4\pi\epsilon_0}\left(\frac{1}{4.6098} - \frac{1}{5.408}\right)$$

$$= \underline{288.14V}$$

(d) The vectors $\bar{R}_1 = (3-0)\bar{a}_x + (4-0.5)\bar{a}_z = 3\bar{a}_x + 3.5\bar{a}_z$

$\bar{R}_2 = (3-0)\bar{a}_x + (4+0.5)\bar{a}_z = 3\bar{a}_x + 4.5\bar{a}_z$

∴ In cartesian coordinate system, the electric field intensity at the point P due to charges at points (0,0,0.5) and (0,0,−0.5) is

$$\bar{E} = \frac{Q_1}{4\pi\epsilon_0 |R_1|^2}\bar{a}_{R_1} + \frac{Q_2}{4\pi\epsilon_0 |R_2|^2}\bar{a}_{R_2}$$

$$= \frac{10^{-6}}{4\pi\epsilon_0}\left[\frac{1}{4.61^3}(3\bar{a}_x + 3.5\bar{a}_z) - \frac{1}{5.41^3}(3\bar{a}_x + 4.5\bar{a}_z)\right]$$

$$= 105.03\bar{a}_x + 65.72\bar{a}_z \text{ V/m}$$

Its magnitude is $|\bar{E}| = \sqrt{105.05^2 + 65.72^2} = 123.89$ V/m.

Compute the torque for a dipole comprising 1 μC charges in an electric field $\bar{E} = 10^3(3\bar{a}_x - \bar{a}_y - \bar{a}_z)$ separated by 1 mm and located on the z-axis at origin.

Q = 1 μC   d = 1 mm $E = 10^3(3\bar{a}_x - a_y - \bar{a}_z)$

$\bar{T} = \bar{P} \times \bar{E}$

$\bar{P} = \overset{Qd\,\bar{a}_z}{1\times10^{-6}\times1\times10^{-3}}\,\bar{a}_z = 10^{-9}\bar{a}_z$

$\bar{T} = 10^{-9}\bar{a}_z \times 10^{3}(3\bar{a}_x - \bar{a}_y - \bar{a}_z)$

$$= 10^{-6}\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 0 & 0 & 1 \\ 3 & -1 & -1 \end{vmatrix}$$

$$= 10^{-6}[\bar{a}_x + 3\bar{a}_y] \text{ N-m.}$$

Calculate the potential due to a dipole of dipole moment 45×10^{-10} C-m at a distance 1 mt from it on its (i) axis, (ii) perpendicular bisector

May 2011, Nov 2

Sol:- (i) $V = \frac{P\cos\theta}{4\pi\epsilon_0 r^2} = \frac{45\times10^{-10}\cos 0^\circ}{4\pi\epsilon_0 (1)^2}$ (∵ θ = 0 on its axis)

$= 40.5$ volts

(ii) at perpendicular bisector θ = 90°

∴ cos θ = 0°

∴ V = 0 volts.

For a dipole $10^{-6}\bar{a}_z$ C-m at the origin in free space, find the potential at a point A(10, π/3, π/2)

Nov 2008

$\bar{P} = 10^{-6}\bar{a}_z$ C-m at (0,0,0)

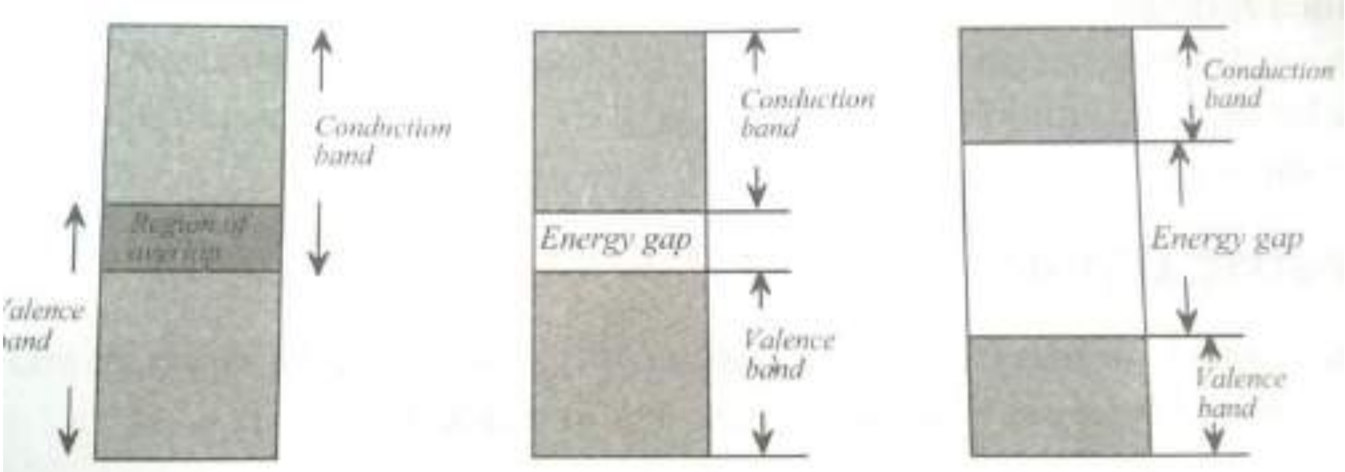
$$\therefore V = \frac{P\cos\theta}{4\pi\epsilon_0 r^2} = \frac{10^{-6}\cos\pi/2}{4\pi\epsilon_0 (10)^2} = \underline{45V}$$

Classification of Materials

Materials are classified as three types, in a broad sense, materials may be classified in terms of their *conductivity* ***σ,*** in mhos per meter or Siemens per meter (S/m), as conductors and nonconductors, or technically as metals and insulators (or dielectrics). The conductivity of a material usually depends on temperature and frequency.

- **Conductors or Metal (***high conductivity* ($\sigma >> 1$))
- **Semi-Conductors (***conductivity in between*)
- **Insulators or Dielectrics (***low conductivity* ($\sigma << 1$))

Materials can also be classified on the basis of their energy bands. The energy bands of materials are shown in the below figure.



(a) Conductors (b) Semiconductor (b) Insulators

Microscopically, the major difference between a conductor and an insulator lies in the amount of free electrons available for conduction of current and energy gap in the materials. These free electrons have to jump from valence band to conduction band through the energy gap. Dielectric materials with huge energy gap and have few electrons available for conduction of current in contrast to conductors, which have an abundance of free electrons with overlapped energy gap.

Dielectric Materials: Types

Dielectric Materials can be classified based on Polarization that takes place in the material under the influence of Electric Field. They are classified as:

- Polar and
- Non-Polar Dielectrics

1. In polar dielectric materials, the molecules or atoms possess a permanent dipole which is randomly oriented as shown in figure (a). Each pair of charges acts as a dipole with the same dipole moment. Since the dipoles are randomly oriented, the net dipole moment is zero. Under the influence of **E**, the dipoles experience a torque and get oriented along the direction of the field as shown in figure (b). This process is called the polarization of polar dielectric materials. (Examples: Water, Sulphur-di-oxide etc.,)
2. The atoms in non-polar materials do not form dipoles until the application of **E**. Under the influence of **E**, the separation of charges comes in existence and they form dipoles with some dipole moment. This process is called polarization of non-polar materials. 9Examples: Hydrogen, Oxygen etc.,)

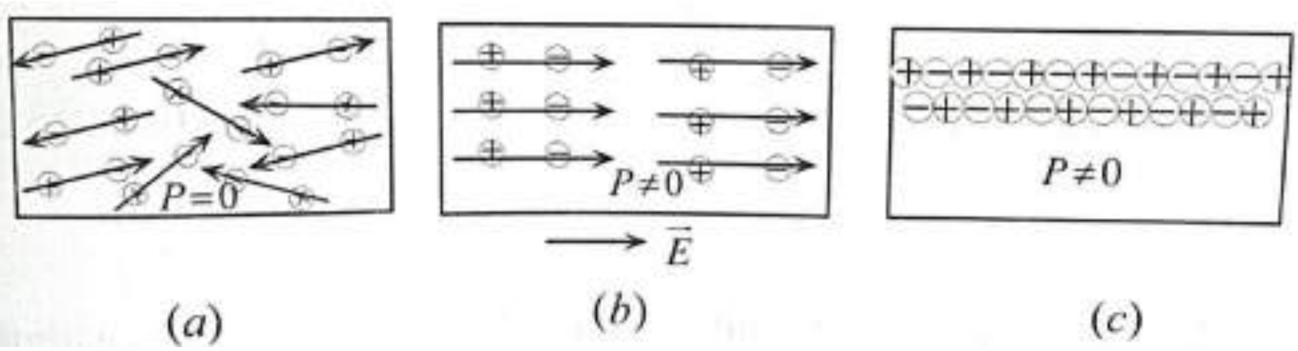


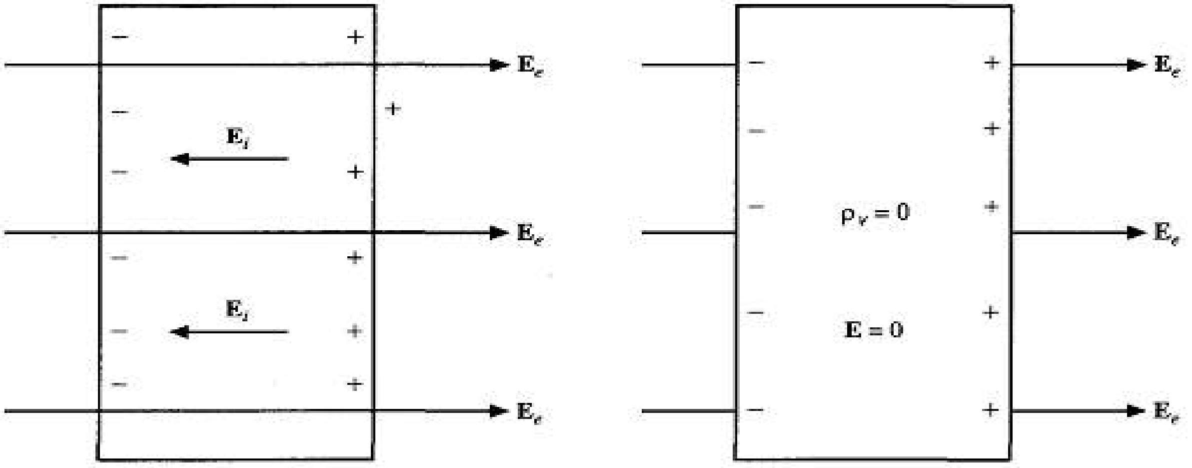
Figure. Polarization of Dielectric Material: (a) polar with no **E**, (b) polar with **E**, and (c) non-polar with **E**.

A dipole results from the displacement of the charges and the dielectric is said to be polarized.

Electric Field in Material Space: Conductors

A conductor has abundance of charge that is free to move. Consider an isolated conductor, such as shown in Figure (a). When an external electric field $\mathbf{E_e}$ is applied, the positive free charges are pushed along the same direction as the applied field, while the negative free charges move in the opposite direction. This charge migration takes place very quickly. The free charges do two things. First, they accumulate on the surface of the conductor and form an *induced surface charge.* Second, the induced charges set up an internal induced field $\mathbf{E_i}$, which cancels the externally applied field $\mathbf{E_e}$. The result is illustrated in Figure (b). This leads to an important property of a conductor:

- A **perfect conductor** cannot contain an electrostatic field within it.
- A conductor is called an *equipotential* body, implying that the potential is the same everywhere in the conductor. This is based on the fact that $\mathbf{E} = -\nabla V = 0$.
- According to Gauss's law, if $\mathbf{E} = 0$, the charge density ρ_v must be zero.



(a) (b)

We conclude again that a perfect conductor cannot contain an electrostatic field within it. Under static conditions:

$$\boxed{\mathbf{E} = 0, \qquad \rho_v = 0, \qquad V_{ab} = 0 \qquad \text{inside a conductor}}$$

Electric Field in Material Space: Dielectrics

To understand the macroscopic effect of an electric field on a dielectric, consider an atom of the dielectric as consisting of a negative charge - *Q* (electron cloud) and a positive charge *+Q* (nucleus) as in Figure (a). Since we have equal amounts of positive and negative charge, the whole atom or molecule is electrically neutral. When an electric field E is applied, the positive charge is displaced from its equilibrium position in the direction of E by the force F + = *QE* while the negative charge is displaced in the opposite direction by the force F_ = *QE.* In the polarized state, the electron cloud is distorted by the applied electric field E. This distorted charge distribution is equivalent, by the principle of superposition, to the original distribution plus a dipole whose moment is

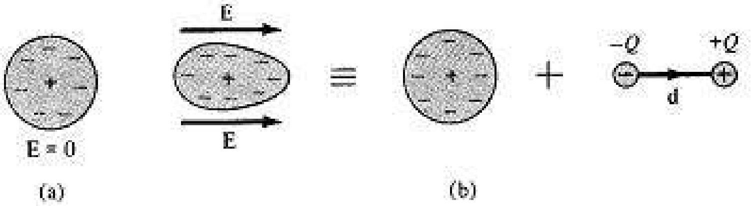
$$\mathbf{p} = Q\mathbf{d}$$

where d is the distance vector from —*Q* to *+Q* of the dipole as in Figure (b). If there are *N* dipoles in a volume Δv of the dielectric, the total dipole moment due to the electric field is

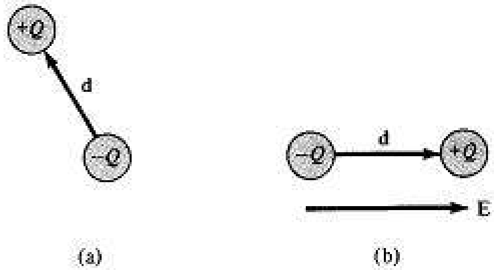
$$Q_1\mathbf{d}_1 + Q_2\mathbf{d}_2 + \cdots + Q_N\mathbf{d}_N = \sum_{k=1}^{N} Q_k\mathbf{d}_k$$

As a measure of intensity of the polarization, we define *polarization* P (in coulombs/meter square) as the dipole moment per unit volume of the dielectric; that is,

$$\boxed{\mathbf{P} = \frac{\lim\limits_{\Delta v \to 0} \sum\limits_{k=1}^{N} Q_k\mathbf{d}_k}{\Delta v}}$$



Polarization of nonpolar atom



Polarization of polar atom

DIELECTRIC CONSTANT

In ferroelectric materials the relationship between $\bar{P}$ & $\bar{E}$ not only is non-linear but also shows hysteresis effects; that is, the polarization produced by a given electric field intensity depends on the past history of the sample.

In some dielectrics $\bar{P}$ is proportional to the applied field $\bar{E}$ as

$$\bar{P} = \chi_e \epsilon_0 \bar{E}$$

where χ (chi) is a dimensionless quantity called "electric susceptib of the material.

$$\therefore \ \bar{D} = \epsilon_0\bar{E} + \chi_e\epsilon_0\bar{E} = (\chi_e+1)\,\epsilon_0\bar{E}$$

$$\boxed{\bar{D} = \epsilon_r\epsilon_0\bar{E} = \epsilon\bar{E}}$$

This relationship depend on the dielectric materia present.

where $\epsilon = \epsilon_0\epsilon_r$

and $\epsilon_r = 1+\chi_e = \frac{\epsilon}{\epsilon_0}$

where ϵ — permittivity of the dielectric F/m

ϵ_0 — permittivity of the free space $= 10^{-9}/36\pi$ F/m

ϵ_r — dielectric constant or relative permittivity

ϵ_r & χ_e are dimensionless

Dielectric Breakdown/Strength

No conductor is ideal It will have some resistance. Similarly

The theory of dielectrics we have discussed so far assumes ideal dielectric. Practically, no dielectric is ideal. When the electric field in a dielectric is sufficiently large, it begins to pull electrons completely out of the molecules, and the dielectric becomes conducting. This phenomenon is called Dielectric Breakdown.

The dielectric breakdown occurs in all kinds of dielectric material (gases, liquids, or solids) and depends on the nature of the materials temperature, humidity and the amount of time that the field is applied.

The minimum value of the ~~di~~electric field at which dielectric breakdown occurs is called the dielectric strength of the dielectric material.

The dielectric strength is the maximum electric field that a dielectric can tolerate or withstand without electrical breakdown

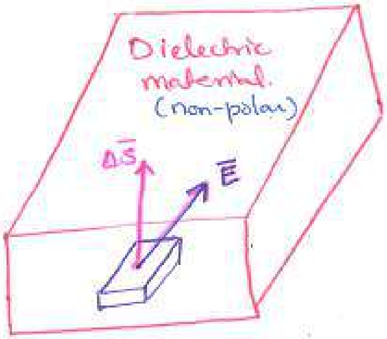
$\overline{D}$ (or) Electric Displacement Vector in Dielectrics.

Let us show that the bound-volume charge density acts like the free-volume charge density in producing an external field. (we shall obtain a result similar to Gauss' Law)

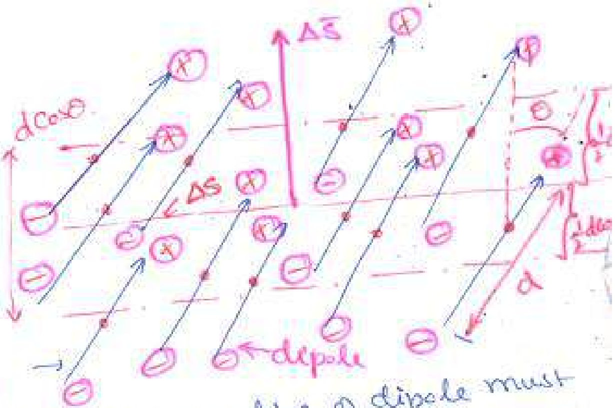
Let us assume that we have a dielectric containing non polar molecule. No molecule has a dipole moment,

$$\overline{P} = 0$$

throughout the material.



Somewhere in the interior of the dielectric let us select an incremental surface element $\Delta\overline{S}$ and apply an electric field $\overline{E}$.



The electric field produces a moment $\overline{p}$ in each molecule

$$\overline{p} = Q\overline{d}$$

such that $\overline{p}$ & $\overline{d}$ makes an angle θ with $\Delta\overline{S}$ as indicated in the figure. The charge is

$\therefore \Delta Q$ ~~Q~~ $Q\,d\cos\theta\,\Delta S$

Each of the charge associated with the creation of dipole must have moved a distance $\frac{1}{2}d\cos\theta$ in the direction perpendicular to $\Delta\overline{S}$.

Therefore, since there are n molecules/m³, the net total charge which crosses the elemental surface in the upward direction is equal to $nQd\cos\theta\,\Delta S$ (or)

$$\boxed{\Delta Q_b = nQ\overline{d}.\Delta\overline{S}}$$

where Q_b — bound charge and not the free charge.

Interms of polarization, the above equation can be written as

$$\boxed{\Delta Q_b = \overline{P}.\Delta\overline{S}}$$

If $\Delta\overline{S}$ is an element of a closed surface inside the dielectric mate then in the direction of $\Delta\overline{S}$ is outward and the net increase in the bound charge within the closed surface is obtained through the integral

$$\boxed{Q_b = -\oint_s \overline{P}.d\overline{S}}$$

-ve sign indicates that the charge is crossing the surfac ΔS

This last relationship has some resemblance to Gauss' law, Now let the total charge enclosed is Q_T,

$$Q_T = \oint_S \epsilon_0 \bar{E} \cdot d\bar{s}$$

where $Q_T = Q_b + Q$ bound charge + free charge enclosed by the surface S.

$$\therefore \boxed{Q = Q_T - Q_b}$$

$$= \oint_S \epsilon_0 \bar{E} \cdot d\bar{s} - \left(-\oint_S \bar{P} \cdot d\bar{s}\right)$$

$$= \oint_S \epsilon_0 \bar{E} \cdot d\bar{s} + \oint_S \bar{P} \cdot d\bar{s}$$

$$Q = \oint_S (\epsilon_0 \bar{E} + \bar{P}) \cdot d\bar{s} \longrightarrow ①$$

Comparing with Gauss' law " the net electric flux penetrating a closed surface is equal to the total charge enclosed ie.

$$\psi = Q_{encl} = \oint_S \bar{D} \cdot d\bar{s} \longrightarrow ②$$

Comparing equations ① and ②

$$\boxed{\epsilon_0 \bar{E} + \bar{P} = \bar{D}} \longrightarrow ③$$

→ Thus when an electric fie is applied, the dielectric material gets polarized.

Now applying divergence to equation ③

Due to polarizat electric flux densi increases.

Volume Charge Densities.

$$\boxed{\begin{aligned} \nabla \cdot \epsilon \bar{E} + \nabla \cdot \bar{P} &= \nabla \cdot \bar{D} = \rho_v \\ \therefore \nabla \cdot \bar{P} &= -\rho_b \\ \nabla \cdot \epsilon \bar{E} &= \rho_T \end{aligned}}$$

∴ Total volume charge density

$$\boxed{\rho_T = \rho_b + \rho_v}$$ (bound volume charge + Volume density of free charg

From eq ③ we conclude that the ~~net~~ application of $\bar{E}$ to the dielectric material causes the flux density $\bar{D}$ to be greater by $\bar{P}$ than it would be in free space.

$\epsilon_0 \bar{E} = \bar{D}$
ut in dielectrics
$\epsilon_0 \bar{E} + \bar{P} = \bar{D}$

In an isotropic material the vectors $\bar{E}$ & $\bar{P}$ are always parallel regardless of the orientation of the field. Although most engineering dielectrics are linear for moderate-to-large field strengths and are also isotropic; single crystals may be anisotropic.

The periodic nature of crystalline materials causes dipole moments to be formed most easily along the crystal axes, and not necessarily in the direction of the applied field.

BOUNDARY CONDITIONS

Sofar, we have considered the existence of the electric field in a homogeneous medium. If the field exists in a region consisting of two differen media, the conditions that the field must satisfy at the interface separating (boundary) the mediams are called the boundary conditions. These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known.

Depending on the nature of the media, there are ~~two~~ three (3) situations

① Boundary conditions between Conductor & free Space

② Boundary condition between Conductor & Dielectric

③ Boundary condition between Dielectric (ϵ_{r1}) & Dielectric (ϵ_{r2}

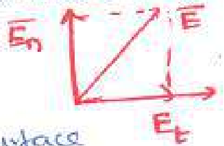
To determine the boundary conditions, we need to use Maxwell's equations & Gauss' Law

$$\boxed{\begin{aligned} &\oint \bar{E}\cdot d\bar{l} = 0 \\ \text{and} \quad &\oint_S \bar{D}\cdot d\bar{s} = Q_{encl} \end{aligned}} \quad \text{or} \quad \oint_S \epsilon \bar{E}\cdot d\bar{s} = Q_{encl}$$

where Q_{encl} is the free charge enclosed by the surface S.

Also we need to decompose the EFI ($\bar{E}$) into two orthogonal components

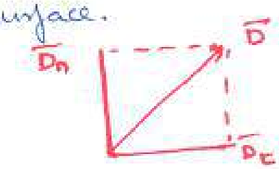
$$\boxed{\bar{E} = \bar{E}_t + \bar{E}_n}$$



$\bar{E}_t$ — the component tangential to the surface

$\bar{E}_n$ — the component normal to the surface.

Similarly $$\boxed{\bar{D} = \bar{D}_t + \bar{D}_n}$$



Note! For an Ideal conductor

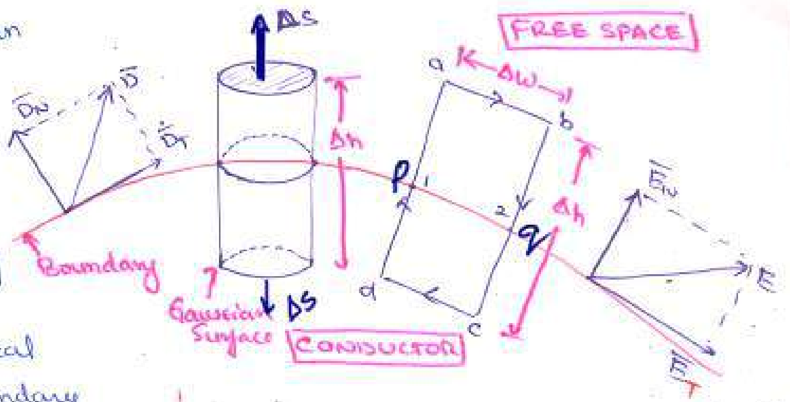
1. The EFI inside a conductor is zero and flux density inside is ze

2. No charge can exist within a conductor. The charge appears on the surface in the form of surface charge density

3. The charge density within the conductor is zero.

Thus $\bar{E}$, $\bar{D}$ & ρ_v within the conductor are zero. While ρ_s is the surface charge density on the surface of the conductor.

CONDUCTOR - FREE SPACE BOUNDARY CONDITIONS



Consider a boundary between conductor & free space. The Conductor is ideal having infinite conductivity.

To determine the boundary conditions let us use a closed path and cylindrical gaussian surface at the boundary

Consider a rectangular closed path 'abcda'. Applying Maxwell's Equation $\oint \bar{E}\cdot d\bar{l} = 0$

$$\oint \bar{E}\, d\bar{l} = \int_a^b \bar{E}\cdot d\bar{l} + \int_b^c \bar{E}\cdot dl + \int_c^d \bar{E}\cdot dl + \int_d^a \bar{E}\cdot dl$$

remembering $\bar{E}$ inside the conductor is zero. And the rectangle is placed in such a way that, half of it is in the conductor and $(\Delta h/2)$ remaining is in the free space.

$$\therefore \oint \bar{E}\, d\bar{l} = E_T \Delta w - \left(E_{Nq}\tfrac{1}{2}\Delta h + \underbrace{E_{Nq}\tfrac{1}{2}\Delta h}_{=0}\right)$$

$$0 + \left(E_{Np}\tfrac{1}{2}\Delta h + \underbrace{E_{Np}\tfrac{1}{2}\Delta h}_{=0}\right) = 0$$

since $\int_c^d \bar{E}\cdot dl = 0$ inside the conductor $\bar{E} = 0$.

$$\therefore \oint \bar{E}\cdot d\bar{l} = E_T \Delta w - E_{Nq}\tfrac{1}{2}\Delta h + \underset{=0}{E_{Np}\tfrac{1}{2}\Delta h}$$

As $\Delta h \to 0$ keeping Δw small but finite

$$\oint \bar{E}\, d\bar{l} = E_T \Delta w = 0$$

$$\therefore \boxed{E_T = 0}$$

Thus the tangential component of the electric field intensity is zero at the boundary between the conductor & free space.

Similarly $\bar{D} = \epsilon_0 \bar{E}$

$$\therefore \bar{D}_T = \epsilon_0 \bar{E}_T = 0$$

$$\therefore \boxed{\bar{D}_T = 0}$$

Thus the tangential component of $\bar{D}$ is zero at the boundary

The condition on the normal field is found by considering D_N rather than E_N and choosing a small cylinder as the gaussian surface.

Using Gauss' Law $\quad \oint_{top} \bar{D}\cdot d\bar{s}$

$$\oint \bar{D}\cdot d\bar{s} = Q_{encl}$$

$$= \int_{top} \bar{D}\cdot d\bar{s} + \int_{bottom} \bar{D}\cdot d\bar{s} + \int_{lateral} \bar{D}\cdot d\bar{s} = Q_{en}$$

$\int_{bottom} \bar{D}\cdot ds = 0$ since $\bar{D} = 0$ inside the conduc

$\int_{lateral} \bar{D}\cdot d\bar{s} = 0$ since surface area $ds = 2\pi\rho\Delta h$ as $\Delta h \to 0$; $ds = 0$

$$\therefore \oint \bar{D}\cdot d\bar{s} = \int_{top} \bar{D}\cdot d\bar{s} = Q_{encl}$$

$$= D_N \int_{top} d\bar{s} = D_N \Delta S = \rho_s \Delta S$$

$$\boxed{\therefore D_N = \rho_s = \epsilon_0 E_N}$$

Thus the flux leaves the surface normally and the normal component of the flux density is equal to the surface charge density

Summary!

1. The static EFI inside a conductor is zer
2. The static EFI at the surface of a conductor is everywhere directed norm to that surface.
3. The conductor surface is equipotential surface

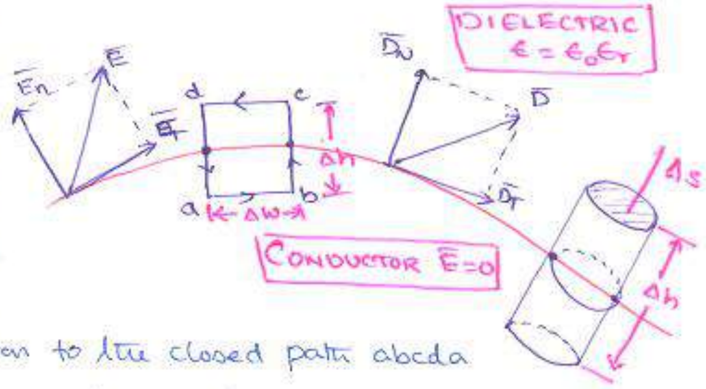
$$E_T = D_T = 0$$

$$D_N = \rho_s$$

$$E_N = \frac{\rho_s}{\epsilon_0} = \frac{D_N}{\epsilon_0}.$$

CONDUCTOR - DIELECTRIC BOUNDARY CONDITIONS

To determine the boundary conditions for a conductor - dielectric interface, the same procedure used for dielectric - dielectric interface except that we incorporate the fact that ($E = 0$) inside the conductor.



Applying Maxwells equation to the closed path abcda

$$\oint \bar{E}.d\bar{l} = \int_a^b \bar{E}.d\bar{l} + \int_b^c \bar{E}.d\bar{l} + \int_c^d \bar{E}.d\bar{l} + \int_d^a \bar{E}.d\bar{l} = 0$$

$$= \underset{\text{inside}}{\underline{0.\Delta w}} + \left(\underset{\text{inside}}{\underline{0.\frac{\Delta h}{2}}} + \cancel{E_N\frac{\Delta h}{2}}\right) - E_T.\Delta w - \left(\cancel{E_N\frac{\Delta h}{2}} + \underset{\text{inside}}{\underline{0.\frac{\Delta h}{2}}}\right) = 0$$

$$\bar{E}_T\Delta w = 0$$

As $\Delta h \to 0$

$$\boxed{E_T = 0} \qquad \boxed{E_T = 0}$$

Similarly, by applying Gauss's law to the small cylinder as the gaussian surface

$$\oint \bar{D}.d\bar{s} = Q_{encl}$$

$$= \int_{top} \bar{D}.d\bar{s} + \int_{bottom} \bar{D}.d\bar{s} + \int_{lateral} \bar{D}.d\bar{s} = Q_{encl}$$

As $\Delta h \to 0$,

$$\oint \bar{D}.d\bar{s} = Q_{encl} = \int D_N.\Delta S - \underset{\substack{\text{Inside}\\ \bar{D}=0\\ \epsilon E=0}}{\underline{0.\Delta S}} + \underset{\substack{\Delta S=2\pi r\Delta h\\ \Delta h\to 0}}{\underline{D_T.0}}$$

$$\text{(or)} \quad \Delta Q = D_N.\Delta S$$

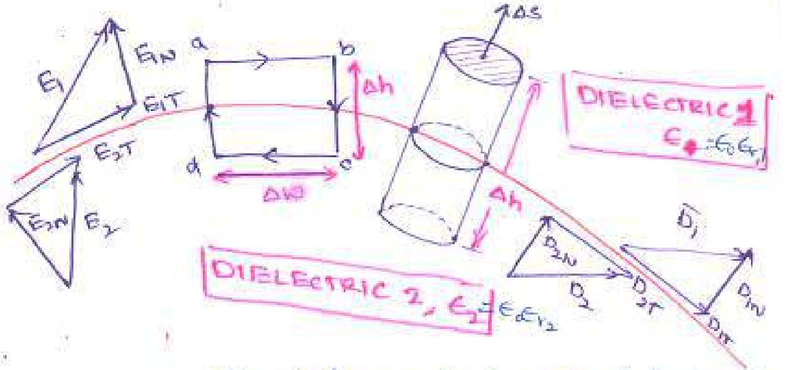
$$D_N = \frac{\Delta Q}{\Delta S} = \rho_s$$

$$\text{or} \quad \boxed{D_N = \rho_s}$$

Thus if the boundary is between conductor & dielectric with $\epsilon = \epsilon_0\epsilon_r$ then

$$\boxed{\begin{array}{l} E_T = D_T = 0 \\ D_N = \rho_s \\ \text{and} \quad E_N = \dfrac{\rho_s}{\epsilon} = \dfrac{\rho_s}{\epsilon_0\epsilon_r} \end{array}}$$

DIELECTRIC - DIELECTRIC BOUNDARY CONDITIONS



Consider the $\bar{E}$ field existing in the region of two different dielectrics characterized by $\epsilon_1 = \epsilon_0 \epsilon_{r1}$ and $\epsilon_2 = \epsilon_0 \epsilon_{r2}$

The fields in the media ① & 2. $\bar{E}_1$ & $\bar{E}_2$ respectively can be decomposed as

$$\bar{E}_1 = \bar{E}_{1T} + \bar{E}_{1N}$$

$$\bar{E}_2 = \bar{E}_{2T} + \bar{E}_{2N}$$

Now applying Maxwells equation $\oint \bar{E} \cdot d\bar{l} = 0$ to the closed path abcda, we obtain

$$\oint \bar{E} \cdot d\bar{l} = \int_a^b \bar{E} \cdot d\bar{l} + \int_b^c \bar{E} \cdot d\bar{l} + \int_c^d \bar{E} \cdot d\bar{l} + \int_d^a \bar{E} \cdot d\bar{l}$$

$$0 = E_{1T} \Delta w - E_{1N} \frac{\Delta h}{2} - E_{2N} \frac{\Delta h}{2} - E_{2T} \Delta w + E_{2N} \frac{\Delta h}{2} + E_{1N} \frac{\Delta h}{2}$$

As $\Delta h \rightarrow 0$

$$0 = E_{1T} \Delta w - E_{2T} \Delta w$$

$$= (E_{1T} - E_{2T}) \Delta w$$

$$\Rightarrow \boxed{E_{1T} = E_{2T}}$$

Thus the tangential components of EFI at the boundary in both the dielectrics remain same i.e. electric field intensity is continuous across the boundary.

Since $\bar{D} = \epsilon \bar{E} = \bar{D}_T + \bar{D}_N$ can be written as

$$\frac{D_{1T}}{\epsilon_1} = E_{1T} = E_{2T} = \frac{D_{2T}}{\epsilon_2}$$

or $$\boxed{\frac{D_{1T}}{\epsilon_1} = \frac{D_{2T}}{\epsilon_2}}$$

Thus the tangential component of $\bar{D}$ undergoes some change across the interface hence tangential $\bar{D}$ is said to be discontinuous across the boundary.

Similarly to find normal components we apply cylindrical Gaussian surface

Now applying Gauss law

$$\oint \bar{D} \cdot d\bar{s} = Q_{encl}$$

$$= \int_{top} \bar{D} \cdot d\bar{s} + \int_{bottom} \bar{D} \cdot d\bar{s} + \int_{lateral} \bar{D} \cdot d\bar{s}$$

$$= \int_{top} \bar{D} \cdot d\bar{s} + \int_{bottom} \bar{D} \cdot d\bar{s} + 0$$

as $\Delta h \rightarrow 0$ $ds = 2\pi r \Delta h \rightarrow 0$

$$\therefore \Delta Q = D_{1N} \Delta S - D_{2N} \Delta S = \rho_s \Delta S$$

$$\boxed{\therefore D_{1N} - D_{2N} = \rho_s}$$

ρ_s - is free charge density placed deliberately at the boundary.

If no charge exist at the surface (in perfect dielectric - no free charge exist) then $\rho_s = 0$

$$\boxed{\therefore D_{1N} = D_{2N}}$$

Hence the normal component of flux density $\bar{D}$ is continuous at the boundary between the two perfect dielectrics.

Now $D_{N1} = \epsilon_1 E_{N1}$; $D_{N2} = \epsilon_2 E_{N2}$

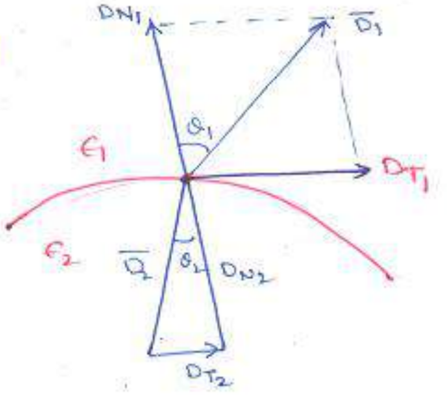
$$\therefore \frac{D_{N1}}{D_{N2}} = \frac{\epsilon_1}{\epsilon_2} \frac{E_{N1}}{E_{N2}} = 1$$

as $D_{1N} = D_{2N}$

$$\therefore \boxed{\frac{E_{N1}}{E_{N2}} = \frac{\epsilon_2}{\epsilon_1} = \frac{\epsilon_{r2}}{\epsilon_{r1}}}$$

Thus the normal component of the $\bar{E}$ are inversely proportional to the relative permittivities of the two media.

REFRACTION of $\overline{D}$ & $\overline{E}$ at the BOUNDARY



The directions of $\overline{D}$ & $\overline{E}$ change at the boundary between the two dielectrics.

$\overline{D_1} = \epsilon_1 \overline{E_1}$ & $\overline{D_2} = \epsilon_2 \overline{E_2}$

$|\overline{D_1}| = D_1$ & $|\overline{D_2}| = D_2$

$\therefore D_{N_1} = D_1 \cos\theta_1$

$D_{N_2} = D_2 \cos\theta_2$

but $D_{N_1} = D_{N_2}$

$\therefore D_1 \cos\theta_1 = D_2 \cos\theta_2 \Rightarrow \frac{D_{N_1}}{D_{N_2}} = \frac{D_1 \cos\theta_1}{D_2 \cos\theta_2} \rightarrow ①$

while $\frac{D_{T_1}}{D_{T_2}} = \frac{\epsilon_1}{\epsilon_2}$

But $D_{T_1} = D_1 \sin\theta_1$

$D_{T_2} = D_2 \sin\theta_2$

$\therefore \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\epsilon_1}{\epsilon_2} = \frac{D_{T_1}}{D_{T_2}} \rightarrow ②$

dividing eq ② by eq (1)

$$\frac{D_1 \sin\theta_1 / D_1 \cos\theta_1}{D_2 \sin\theta_2 / D_2 \cos\theta_2} = \frac{D_{T_1} / D_{N_1}}{D_{T_2} / D_{N_2}} = \frac{\epsilon_1}{\epsilon_2}$$

$$= \frac{\tan\theta_1}{\tan\theta_2} = \frac{D_{T_1}}{D_{T_2}} = \frac{\epsilon_1}{\epsilon_2}$$

since $D_{N_1} = D_{N_2}$

$$\boxed{\therefore \frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2} = \frac{\epsilon_{r_1}}{\epsilon_{r_2}}}$$

Law of Refraction

Angles θ_1 & θ_2 are dependent on permittivities of two media and not on $\overline{D}$ or $\overline{E}$

Thus if $\epsilon_1 > \epsilon_2$ then $\theta_1 > \theta_2$.

The magnitude of $\overline{D}$ in the region② can be obtained as

$$D_2^2 = D_{N_2}^2 + D_{T_2}^2 = (D_1 \cos\theta_1)^2 + D_{T_2}^2$$

Now $D_{T_2} = D_2 \sin\theta_2 = D_1 \sin\theta_1 \frac{\epsilon_2}{\epsilon_1}$

$$\therefore D_2^2 = (D_2 \cos\theta_1)^2 + \left(D_1 \sin\theta_1 \frac{\epsilon_2}{\epsilon_1}\right)^2$$

$$\therefore \boxed{D_2 = D_1 \sqrt{\cos^2\theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2\theta_1}}$$

Similarly

$$E_2 = E_1 \sqrt{\sin^2\theta_1 + \left(\frac{\epsilon_1}{\epsilon_2}\right)^2 \cos^2\theta_1}$$

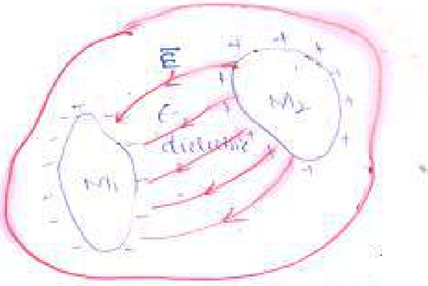
The equation shows that

1. $\overline{D}$ is larger in the region of larger permittivity
2. $\overline{E}$ is larger in the region of smaller permittivity
3. $|\overline{D_1}| = |\overline{D_2}|$ if $\theta_1 = \theta_2 = 0°$
4. $|\overline{E_1}| = |\overline{E_2}|$ if $\theta_1 = \theta_2 = 90°$

Boundary conditions allow us to find quickly the field on one side of a boundary if we know the field on the other side.

CAPACITANCE

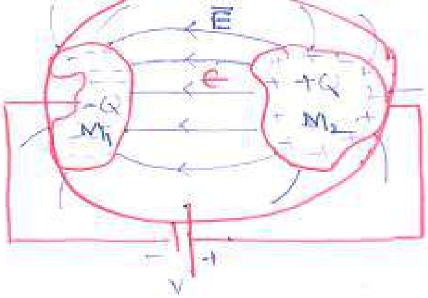
Now let us consider two conductors embedded in a homogeneous dielectric. Conductors M_1 & M_2 carries charges Q equal in magnitude but in opposite polarity.



Two oppositely charged conductors M_1 & M_2 surrounded by a uniform dielectric.

The potential difference between M_1 & M_2 is

$$V = V_1 - V_2 = -\int_2^1 \bar{E}\cdot d\bar{l}$$



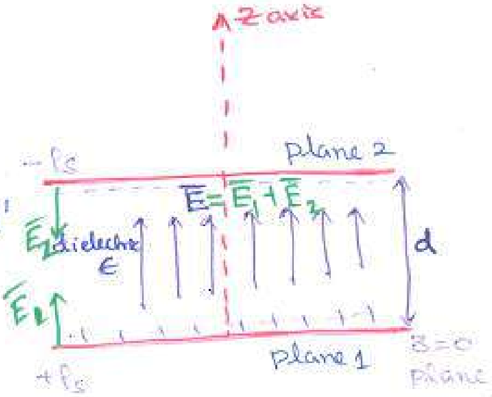
Now we may define the capacitance of this two conductor system as the ratio of the magnitude of the total charge on either conductor to the magnitude of the potential difference between the conductors,

$$\boxed{C = \frac{Q}{V} = \frac{\oint_S \epsilon \bar{E}\cdot d\bar{S}}{-\int_2^1 \bar{E}\cdot d\bar{l}}} \text{ (or) } \frac{\oint_S \epsilon \bar{E}\cdot d\bar{S}}{\int_2^1 \bar{E}\cdot d\bar{l}}$$

−ve sign is removed because interested in absolute value of V.

Note:- The capacitance depends on the physical dimensions of the system and the properties of the dielectric such as permittivity of the dielectric.

① Parallel Plate Capacitor



Consider ~~parallel~~ plate capacitor consists of two metallic plates separated by distance 'd' filled with dielectric of permittivity ∈. Lower plate carries uniform positive charge density ρ_s and upper plate $-\rho_s$.

The plate 1 is placed at $z=0$ & plate 2 is at $z=d$ plane, parallel to xy-plane.

Let the area of the plate is S.

$$\therefore \quad \rho_s = \frac{Q}{S} \ c/m^2$$

Now EFI($\bar{E}$) between the plates $= \bar{E} = \bar{E}_1 + \bar{E}_2$

$$\bar{E}_1 = \frac{\rho_s}{2\epsilon}\bar{a}_N = \frac{\rho_s}{2\epsilon}\bar{a}_z \ V/m \quad \text{plate 1}$$

$$\bar{E}_2 = -\frac{\rho_s}{2\epsilon}\bar{a}_N = -\frac{\rho_s}{2\epsilon}(-\bar{a}_z) \ V/m \quad \text{Plate 2}$$

the direction of $\bar{E}$ is downward ie in $-\bar{a}_z$ direction

In between the plates

$$\overline{E} = \overline{E_1} + \overline{E_2} = \frac{\rho_s}{2\epsilon}\overline{a_z} + \frac{\rho_s}{2\epsilon}\overline{a_z} = \frac{\rho_s}{\epsilon}\overline{a_z}$$

The potential difference is given by

$$V = -\int_2^1 \overline{E}\cdot d\overline{L} = -\int_2^1 \frac{\rho_s}{\epsilon}\overline{a_z}\cdot d\overline{L}$$ – ve sign neglected as we are interested in absolute voltage

Now $d\overline{L} = dx\,\overline{a_x} + dy\,\overline{a_y} + dz\,\overline{a_z}$

$V = \int_2^1 \overline{E}\cdot d\overline{l}$

$$\therefore\ V = \int_{z=d}^{z=0} \frac{\rho_s}{\epsilon}\overline{a_z}\cdot(dx\,\overline{a_x} + dy\,\overline{a_y} + dz\,\overline{a_z})$$

$$V = -\int_{z=d}^{z=0}\frac{\rho_s}{\epsilon}dz = -\frac{\rho_s}{\epsilon}[z]_d^0 = \frac{-\rho_s[-d]}{\epsilon}$$

$$\boxed{V = \frac{\rho_s d}{\epsilon}\,V}$$

∴ The capacitance between the parallel plates is

$$\boxed{C = \frac{Q}{V} = \frac{\rho_s S}{\frac{\rho_s d}{\epsilon}} = \frac{\epsilon S}{d}\,F = \frac{\epsilon_0\epsilon_r S}{d}\,F}\ (\text{or})\ \frac{\epsilon_0\epsilon_r A}{d}\,F$$

S, A – surface area

From the equation, the value of capacitance depends upon

1. The permittivity of the dielectric used
2. The area of cross section of the plates (surface area)
3. The distance of separation of the plates

It does not depend on the charge or the potential difference between the plates.

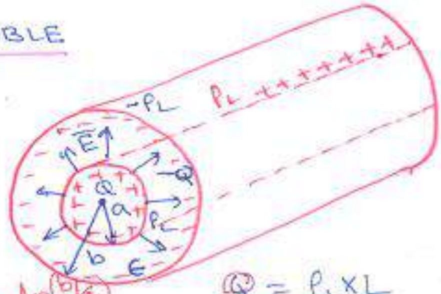
② COAXIAL CAPACITOR (or) COAXIAL CABLE

Consider a coaxial cable of length L of two conductors of inner radius a & outer radius b.

~~$Q = \epsilon\oint E\cdot ds$~~

~~$= \epsilon E_\rho 2\pi\rho L$~~

~~$\therefore \overline{E}$~~



$Q = \rho_L \times L$

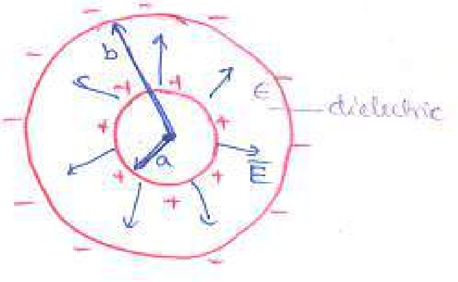
$$\overline{E} = \frac{\rho_L}{2\pi\epsilon r}\overline{a_r}\ \text{or}\ \frac{\rho_L}{2\pi\epsilon\rho}\overline{a_\rho}$$

$$\& \ V = -\int \overline{E}\cdot d\overline{L} = -\int_{r=b}^{a}\frac{\rho_L}{2\pi\epsilon\rho}\overline{a_\rho}\cdot d\rho\,\overline{a_\rho} = \frac{\rho_L}{2\pi\epsilon}\ln(b/a) = \frac{Q}{2\pi\epsilon L}\ln(b/a) \rightarrow \rho_L = Q/L$$

$$\therefore\ \boxed{C = \frac{Q}{V} = \frac{2\pi\epsilon L}{\ln b/a}\ F}$$

$$C = \frac{\cancel{Q}}{\frac{\cancel{Q}}{2\pi\epsilon L}\ln(b/a)} = \frac{2\pi\epsilon L}{\ln(b/a)}\ F.$$

③. SPHERICAL CAPACITOR .



A spherical capacitor is the case of two concentric spherical conductors.

$$\overline{E} = \frac{Q}{4\pi\epsilon r^2}\overline{a}_r \quad V/m$$

ϵ_1

$$V = -\int_{r=b}^{a}\overline{E}.d\overline{l} = -\int_{r=b}^{a}\frac{Q}{4\pi\epsilon r^2}\overline{a}_r.d\overline{l}$$

$$= -\int_{r=b}^{a}\frac{Q}{4\pi\epsilon r^2}\overline{a}_r.dr\,\overline{a}_r \qquad (\because d\overline{l} = dr\,\overline{a}_r)$$

$$= -\int_{r=b}^{a}\frac{Q}{4\pi\epsilon r^2}dr = -\frac{Q}{4\pi\epsilon}\left[-\frac{1}{r}\right]_{r=b}^{a}$$

$$V = \frac{Q}{4\pi\epsilon}\left[\frac{1}{a}-\frac{1}{b}\right] V$$

Now $C = \frac{Q}{V} = \dfrac{Q}{\frac{Q}{4\pi\epsilon}\left[\frac{1}{a}-\frac{1}{b}\right]}$

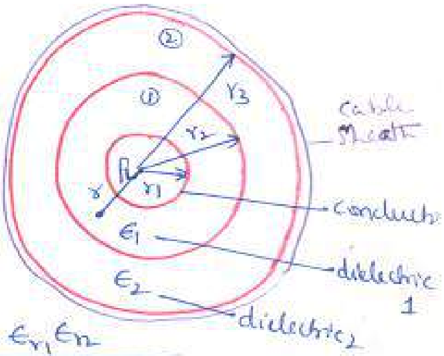
$$\therefore \boxed{C = \frac{4\pi\epsilon}{\left[\frac{1}{a}-\frac{1}{b}\right]} \text{ F.}}$$

④ Capacitance of Single Isolated Sphere

$$C = \frac{4\pi\epsilon}{\left(\frac{1}{a}-\frac{1}{\infty}\right)} \qquad b = \infty$$

$$\therefore \boxed{C = 4\pi\epsilon a \text{ F.}}$$

⑤ Capacitance of Co-axial cable with two dielectrics (composite)



Let ρ_L be the charge density.

$$E_1 = \frac{\rho_L}{2\pi\epsilon_1 r}\ (r_1 \le r \le r_2)$$

$$E_2 = \frac{\rho_L}{2\pi\epsilon_2 r}\ (r_2 \le r \le r_3)$$

If then, the potential differences across the dielectrics ① and ② are V_1 and V_2 respectively,

$$V_1 = -\int_{r_2}^{r_1}E_1 dr = \frac{\rho_L}{2\pi\epsilon_1}\ln\frac{r_2}{r_1}$$

$$V_2 = -\int_{r_3}^{r_2}E_2 dr = \frac{\rho_L}{2\pi\epsilon_2}\ln\frac{r_3}{r_2}$$

potential difference between inner conductor & sheath

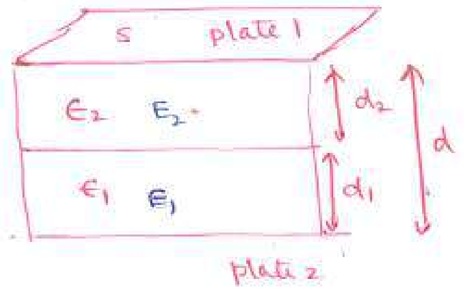
$$V = V_1 + V_2 = \frac{\rho_L}{2\pi\epsilon_0}\left[\frac{1}{\epsilon_{r_1}}\ln\frac{r_2}{r_1} + \frac{1}{\epsilon_{r_2}}\ln\frac{r_3}{r_2}\right]$$

$\therefore$

$$C = \frac{0.0241\,\epsilon_{r_1}\epsilon_{r_2}}{\epsilon_{r_2}\ln\frac{r_2}{r_1} + \epsilon_{r_1}\ln\frac{r_3}{r_2}}$$

$$\boxed{C = \frac{0.0241\,\epsilon_1\epsilon_2}{\epsilon_2\ln\frac{r_2}{r_1} + \epsilon_1\ln\frac{r_3}{r_2}}\ \mu F.}$$

⑥ Composite parallel plate Capacitor



Let Q is the charge on the plate

$\bar{E}_1$ — EFI in region (ϵ_1)

$\bar{E}_2$ — EFI in region (ϵ_2)

Both intensities are uniform

$\therefore V_1 = E_1 d_1$

$V_2 = E_2 d_2$

$\therefore V = V_1 + V_2 = E_1 d_1 + E_2 d_2 \longrightarrow ①$

At a dielectric-dielectric interface, the normal components of flux densities are equal. ie

$D_{N_1} = D_{N_2}$

Now $D_1 = \epsilon_1 E_1$ & $D_2 = \epsilon_2 E_2$

Substituting into ①

$$V = \frac{D_1}{\epsilon_1} d_1 + \frac{D_2}{\epsilon_2} d_2$$

and $\rho_s = D_1 = D_2$

$$\therefore V = \rho_s\left(\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2}\right)$$

$$\therefore C = \frac{Q}{V} = \frac{Q}{\rho_s\left[\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2}\right]}$$

But $Q = \rho_s \times S$

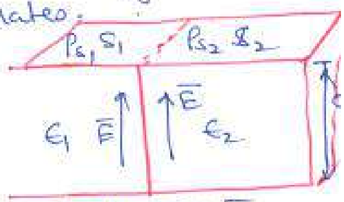
$$\therefore C = \frac{\rho_s . S}{\rho_s\left[\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2}\right]} = \frac{S}{\left[\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2}\right]}$$

$$= \frac{1}{\frac{d_1}{\epsilon_1 S} + \frac{d_2}{\epsilon_2 S}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

$$\therefore \boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}}$$

(or) $C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$

⑦ Dielectric Boundary Normal to the plates.



$E = \frac{V}{d}$

At the boundary, both $\bar{E}_1$ & $\bar{E}_2$ are tangential & for dielectric-dielectric interfaces tangential components are equal

$\therefore E_{T_1} = E_{T_2} = E_1 = E_2 = \frac{V}{d}$

Now $D_1 = \epsilon_1 E_1$ & $D_2 = \epsilon_2 E_2$

$\therefore D_1 = \frac{\epsilon_1 V}{d}$ $D_2 = \frac{\epsilon_2 V}{d}$

On the plate the charge is divided into two parts.

On area S_1 the charge density is $\rho_{s_1} = D_1$ while on area S_2, the charge density is $\rho_{s_2} = D_2$

$Q = Q_1 + Q_2 = \rho_{s_1} S_1 + \rho_{s_2} S_2$

$= D_1 S_1 + D_2 S_2$

$$Q = \frac{\epsilon_1 V_1 S_1}{d} + \frac{\epsilon_2 V S_2}{d}$$

$$\therefore C = \frac{Q}{V} = \frac{\frac{\epsilon_1 V S_1}{d} + \frac{\epsilon_2 V S_2}{d}}{V}$$

$$C = \frac{\epsilon_1 S_1}{d} + \frac{\epsilon_2 S_2}{d} = C_1 + C_2$$

$$\boxed{C_{eq} = C_1 + C_2}$$

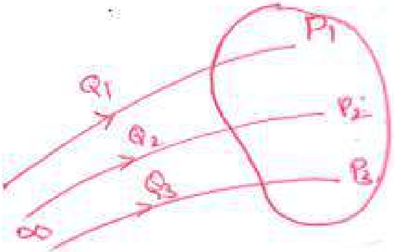
Thus if dielectric boundary is parallel to the plates, the arrangement is equivalent to two capacitors in series. $\frac{1}{C_1}$ $C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$. While if the dielectric boundary is normal to the plates, the arrangement is equivalent to two capacitors in parallel for which $C_{eq} = C_1 + C_2$.

ENERGY DENSITY IN THE ELECTROSTATIC FIELD

Bringing a positive charge from infinity to a point in the field of another positive charge requires work, the work done by external source and energy is exanded. The energy expanded in bringing this charge into position now represents potential energy. [This is analogous to the water lifted at a height h and stored in a tank]

If the external source is removed, it would accelerated away from the fixed charge, aquiring kinetic energy of its own and the capability of doing work.

Inorder to find the PE present in a system of charges, we must find the work done by an external source in positioning the charges.



Consider an empty space where there is no electric field at all. Consider three charges $Q_1, Q_2, \& Q_3$ should be positioned So, No work is required to transfer Q_1 from infinity to P_1 because the space is initially charge free and there is no electric field. [W=0]. Now to place charges Q_2 & Q_3 in positions P_2 & P_3 some work should be done since the charge Q_1 is now presen

Now potential = workdone per unit charge $\left(V = \frac{W}{Q}\right)$

$\therefore$ Workdone $W = V \times Q$

$\therefore$ work done to position Q_2 at $P_2 = V_{21} Q_2$

Q_3 at $P_3 = V_{31} Q_3 + V_{32} Q_3$

due to the field of Q_1 & Q_2

$\therefore$ Total work done in positioning the three charges is

$$W_E = W_1 + W_2 + W_3$$

$$= 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32}) \longrightarrow ①$$

If the charges were positioned in reverse order, then

$$W_E = 0 + Q_2 V_{23} + Q_1 (V_{12} + V_{13}) \longrightarrow ②$$

where V_{23} is the potential at P_2 due to Q_3

adding eq ① & ②

$$2W_E = Q_1 (V_{12} + V_{13}) + Q_2 (V_{21} + V_{23}) + Q_3 (V_{31} + V_{32})$$

$$= Q_1 V_1 + Q_2 V_2 + Q_3 V_3 = \sum_{k=1}^{N} V_k Q_k \text{ J}$$

$Q_k =$

(or) $\boxed{W_E = \frac{1}{2} \sum_{k=1}^{N} V_k Q_k}$ Joules.

If the region has a continuous charge distribution, then,

$$W_E = \frac{1}{2}\int_L \rho_L V\,dl$$ — line charge

$$W_E = \frac{1}{2}\int_S \rho_S V\,ds$$ — surface charge

$$W_E = \frac{1}{2}\int_V \rho_v V\,dv$$ — volume charge.

ENERGY STORED INTERMS OF $\bar{D}$ & $\bar{E}$

Consider the volume charge distribution having uniform charge density ρ_v C/m³.

Hence the total energy stored ~~is~~ ~~the~~ given by

$$W_E = \frac{1}{2}\int_V \rho_v V\,dv$$

according to Maxwells' equation $\rho_v = \nabla\cdot\bar{D}$

$$\therefore W_E = \frac{1}{2}\int_V (\nabla\cdot\bar{D})V\,dv$$

$$= \frac{1}{2}\int_V (\nabla\cdot V\bar{D} - \bar{D}\cdot\nabla V)\,dv$$

∵ if $\bar{A}$ is a vector & V is scalar,

$$\nabla\cdot V\bar{A} = \bar{A}\cdot\nabla V + V(\nabla\cdot\bar{A})$$

(or) $(\nabla\cdot\bar{A})V = \nabla\cdot VA - \bar{A}\cdot\nabla V$

$$\therefore W_E = \frac{1}{2}\int_V \nabla\cdot V\bar{D}\,dv - \frac{1}{2}\int_{vol}\bar{D}\cdot\nabla V\,dv$$

Applying divergence theorem, volume integral can be converted into a closed surface integral.

$$\therefore \cancel{W_E} = \frac{1}{2}\int_V(\nabla\cdot V\bar{D})\,dv = \frac{1}{2}\oint(V\bar{D})\cdot d\bar{s}$$

$$\therefore W_E = \frac{1}{2}\oint(V\bar{D})\cdot d\bar{s} - \frac{1}{2}\int_{vol}(\bar{D}\cdot\nabla V)\,dv$$

Surface integral is equal to zero, for over the closed surface,

$V \rightarrow 0$ as $\frac{1}{r}\rightarrow 0$ $(r\rightarrow\infty)$ &

$\bar{D}\rightarrow 0$ as $\frac{1}{r^2}\rightarrow 0$

$$\therefore W_E = -\frac{1}{2}\int_V \bar{D}\cdot\nabla V\,dv$$

But $\bar{E} = -\nabla V$

$$\therefore W_E = -\frac{1}{2}\int_V \bar{D}\cdot(-\bar{E})\,dv$$

$$\boxed{W_E = \frac{1}{2}\int_V \bar{D}\cdot\bar{E}\,dv \text{ J}}$$

Since $\bar{D} = \epsilon_0\bar{E}$

$$\boxed{W_E = \frac{1}{2}\int_V \epsilon_0 E^2\,dv \text{ J}}$$

$(\because \bar{D}\cdot\bar{E} = \epsilon_0\bar{E}\cdot\bar{E} = \epsilon_0|E|^2$ $\epsilon_0|E||E|)$

$$\boxed{W_E = \frac{1}{2}\int_V \frac{D^2}{\epsilon_0}\,dv \text{ J}}$$

In differential form

$$dW_E = \frac{1}{2}\bar{D}\cdot\bar{E}\,dv$$

$$\Rightarrow \boxed{\frac{dW_E}{dv} = \frac{1}{2}\bar{D}\cdot\bar{E} \text{ J/m}^3}$$

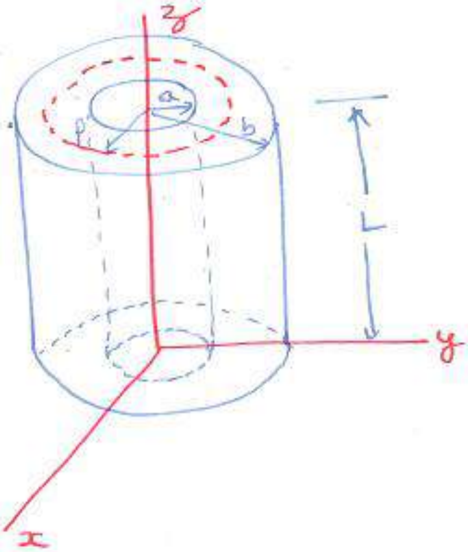
is called Electrostatic Energy Density in the static electric field J/m³

Integrating over the volume gives the total energy present

$$W_E = \int_V\left(\frac{dW_E}{dv}\right)dv$$

Energy Stored in a Co-axial Cable

Consider a coaxial cable with two concentric conducting cylinders of radii a & b (b>a) and length L centred along the z-axis as shown in fig ~~3.13~~. The space between the cable is filled with a dielectric material with permittivity ϵ.



Let Q be the charge on the inner conductor. Also, let the Gaussian surface considered be of radius ρ within the cable.

When $a \le \rho \le b$, since the Gaussian surface encloses the inner conductor, the total charge is

$$Q = \oint_S \bar{D} \cdot d\bar{S} = \oint_S D\,\bar{a}_\rho \cdot \rho\, d\phi\, dz\, \bar{a}_\rho$$

$$= \oint_S \rho D\, d\phi\, dz\, \bar{a}_\rho \cdot \bar{a}_\rho$$

$$= \int_0^L \int_0^{2\pi} \rho D\, d\phi\, dz = 2\pi\rho L D$$

$$\text{(or) } \bar{D} = \frac{Q}{2\pi\rho L}\,\bar{a}_\rho$$

~~$C = \frac{Q}{V} = \frac{2\pi\epsilon L}{\ln b/a}$~~

~~$Q = CV$~~

~~$= \frac{2\pi\epsilon L}{(\ln b/a)} \times 2\pi\epsilon$~~

$Q = \rho_s\, 2\pi a L$

$= \rho_s S$

$S = 2\pi a L$

But we know that the charge density on the inner conductor $\rho_s = \frac{Q}{2\pi a L}$ C/m²

$$\therefore\ \bar{D} = \frac{a\rho_s}{\rho}\,\bar{a}_\rho$$

Energy stored in the electric field between two conductors is

$$W_E = \frac{1}{2}\int_V \frac{D^2}{\epsilon}\, dv = \frac{1}{2\epsilon}\int_V \frac{a^2 \rho_s^2}{\rho^2}\, \rho\, d\rho\, d\phi\, dz$$

$$= \frac{a^2 \rho_s^2}{2\epsilon} \int_0^L \int_0^{2\pi} \int_a^b \frac{1}{\rho}\, d\rho\, d\phi\, dz$$

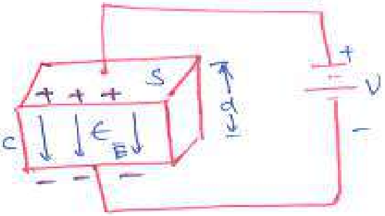
$$= \frac{a^2 \rho_s^2}{2\epsilon}\, 2\pi L \left[\ln\rho\right]_a^b$$

$$\boxed{W_E = \frac{\pi L a^2 \rho_s^2}{\epsilon} \ln\left(\frac{b}{a}\right)} \text{ Joules}$$

Since the charge enclosed in the region $\rho < a$ & $\rho < b$ is zero, the energy stored is zero.

Energy Stored in A Capacitor

$\bar{E} = \frac{V}{d}\bar{a}_N$



Energy stored is

$$W_E = \frac{1}{2}\int_{vol} \bar{D}.\bar{E}\; dv$$

$= \frac{1}{2}\int_{v} \epsilon E.E\, dv =$ but $\bar{E}.\bar{E} = |\bar{E}|^2$

$= \frac{1}{2}\int_{v} \epsilon |E|^2\, dv$ but $|\bar{E}| = \frac{V}{d}$

$= \frac{1}{2}\epsilon \frac{V^2}{d^2}\int_{vol} dv$ $\int_v dv$ = volume = S × d

$= \frac{1}{2}\epsilon \frac{V^2 S d}{d^2} = \frac{1}{2}\frac{\epsilon S}{d} V^2$

$\boxed{W_E = \frac{1}{2} C V^2}$ Joules

Energy stored increases with increase in ε

(a) Find the maximum charge that can be held on the isolated sphere 2m in diameter, the sphere being in air with dielectric strength 40kV/cm

(b) What would be the maximum charge if this sphere is in oil of $\epsilon_r = 3.5$ and dielectric strength of 75 kV/cm? (June 2009)

(a). For sphere $E_1 = \frac{Q}{4\pi\epsilon r^2}$

and for air $\epsilon = \epsilon_0$

∴ The maximum charge is $Q = 4\pi\epsilon_0 E_1 r^2$

$= 4\pi \times 8.854\times10^{-12}\times 40\times10^5 \times 10^2$

$= \underline{1.78 \text{ mC}}$

(b) For oil $\epsilon = \epsilon_0\epsilon_r = 3.5\epsilon_0$

$\therefore Q = 4\pi\times 8854\times10^{-12}\times 3.5\times 75\times10^5\times 2^2 = \underline{11.68 \text{ mC}}$.

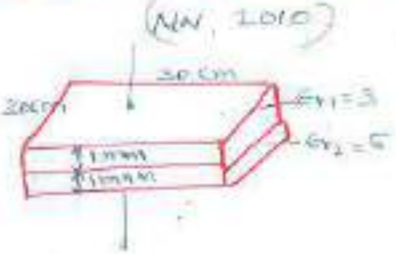
Eg: Obtain the capacitance of an isolated conducting sphere of radiu 1 cm.

For an isolated sphere $C = 4\pi\epsilon_0 r$

$= 4\times\pi\times 8.854\times10^{-12}\times 0.01$

$= 1.11$ pF

A parallel plate capacitor consists of two parallel plates each 30 cm × 30 cm, spaced 2 mm apart and two dielectrics, each 1 mm thick, having relative permittivities of 3 and 5 respectively. If the potential difference between the plates is 5000 V, calculate the voltage gradient in each dielectric. (NW, 2010)



capacitance is given by $C = \frac{C_1 C_2}{C_1 + C_2}$

where $C_1 = \frac{\epsilon_0 \epsilon_{r1} S}{d} = \frac{3\epsilon_0 \times 9 \times 10^{-2}}{10^{-3}} = 270\epsilon_0$

and $C_2 = \frac{\epsilon_0 \epsilon_{r2} S}{d} = \frac{5\epsilon_0 \times 9 \times 10^{-2}}{10^{-3}} = 450\epsilon_0$

$\therefore$ total capacitance $= C = \frac{C_1 C_2}{C_1 + C_2} = 168.75\epsilon_0 = 1.5 \, nF$.

If V_1 & V_2 are potentials across the dielectrics, then

$$V_1 = \frac{V C_2}{C_1 + C_2} = \frac{5000 \times 450}{720} = 3125 \, V$$

$$V_2 = \frac{V C_1}{C_1 + C_2} = \frac{5000 \times 270}{720} = 1875 \, V$$

$\therefore$ Voltage gradient across the first dielectric

$$E_1 = V_1 / d = \frac{3125}{10^{-3}} = 3125 \, kV/m$$

$$E_2 = V_2 / d = \frac{1875}{10^{-3}} = 1875 \, kV/m.$$

For a linear dielectric, show that $\bar{E}_{total} = \bar{E}_{ext} / \epsilon_r$ where ϵ_r is the relative permittivity of the dielectric, $\bar{E}_{total}$ - is the total electric field and $\bar{E}_{ext}$ is the electric field due to charges other than bound charges.

Let $\bar{E}_{in}$ - electric field due to bound charges.

From the theory of polarization

$$\bar{D}_{total} = \epsilon_0 \bar{E}_{total} + \bar{P}$$

$$\bar{P} = \chi_e \epsilon_0 \bar{E}_{total}$$

$$\therefore \bar{D}_{total} = \epsilon_0 (1 + \chi_e) \bar{E}_{total}$$

$$\bar{D}_{total} = \epsilon \bar{E}_{total}$$

Also $\bar{D}_{total} = \epsilon_0 \bar{E}_{ext}$

$$\epsilon \bar{E}_{total} = \epsilon_0 \bar{E}_{ext}$$

$$\cancel{\epsilon_0} \epsilon_r \bar{E}_{total} = \cancel{\epsilon_0} \bar{E}_{ext}$$

$$\bar{E}_{total} = \frac{\bar{E}_{ext}}{\epsilon_r}$$

A parallel plate capacitor with air as dielectric has a plate area of 36π cm² and a separation of 1mm between the plates. It is charged to 100 V by connecting it across a battery. If the battery is disconnected and plate separation is increased to 2mm, calculate the change in (i) potential difference across the plates and (ii) energy stored. May 2011.

Before the separation of the battery

$d_1 = 1\,mm = 10^{-3}\,m$

$S = 36\pi\,cm^2 = 36\pi \times 10^{-4}\,m^2$

ϵ_r for air $= 1$ $\quad V_1 = 100V$

$$\therefore C_1 = \cancel{Q}\,\frac{\epsilon S}{d_1} = \frac{\epsilon_0 \epsilon_r S}{d_1} = \frac{\epsilon_0 \times 36 \times \pi \times 10^{-4}}{10^{-3}}$$

Charge on the capacitor $Q = C_1 V_1 = 100 \times 100 \times 10^{-12} = 10nC$

If the plate separation is increased to 2mm, $d_2 = 2 \times 10^{-3}\,m$

$$\text{then } C_2 = \frac{\epsilon S}{d_2} = \frac{8.854 \times 10^{-12} \times 36\pi \times 10^{-4}}{2 \times 10^{-3}} = 50\,pF.$$

When the battery is disconnected the charge remains same.

(i) potential difference across the plate is

$Q = V_2 C_2$

$$\therefore V_2 = \frac{Q}{C_2} = \frac{10 \times 10^{-9}}{50 \times 10^{-12}} = 200\,V$$

So the potential difference is increased from 100 V to 200V

(ii) Energy stored

when $V_1 = 100V$, $W_1 = \frac{1}{2} C_1 V_1^2 = \frac{1}{2} \times 100 \times 10^{-12} \times 10^2 = 0.5\mu J$

when $V_2 = 200V$ $\quad \omega_1 = \frac{1}{2} C_2 V_2^2 = \underline{0.5\mu J}$

There is no change in energy.

There are three types of Currents

1. **Conduction or Drift Current**; The current flowing due to flow of free electrons or drifting electrons in ~~da~~ conductors under the influence of applied voltage. The direction is opposite to the direction of flow of free electrons.

2. **Convection or Displacement Current:** The current due to the flow of charges in dielectric under the influence of a time varying electric field intensity is called displacement current.

3. **Diffusion Current:** The current flowing due to the movement of free electrons and free holes to make uniform charge density in semiconduc materials is called diffusion current.

Convection current is distinct from conduction current, does not involve conductors &, consequently does not satisfy ohm's law.

CURRENT and CURRENT DENSITY

Electric charge in motion constitute a current. The current can be measured by measuring how many charges are passing through a specified surface or a point in a material per second. The flow of charge per unit time i.e. rate of flow of charge at a specified point or across a specified surface is called an electric current.

$$\boxed{I = \frac{dQ}{dt}}$$

In field theory we are usually interested in events occurring at a point rather than within some large region, and we shall now introduce the concept of Current Density ($\vec{J}$ A/m²).

If current ΔI crossing an incremental surface ΔS, the current densit is

$$J_N = \frac{\Delta I}{\Delta S}$$

or

$$\boxed{\Delta I = J_N \Delta S}$$

assuming that the current density is perpendicular to the su

If the current density is not normal to the surface

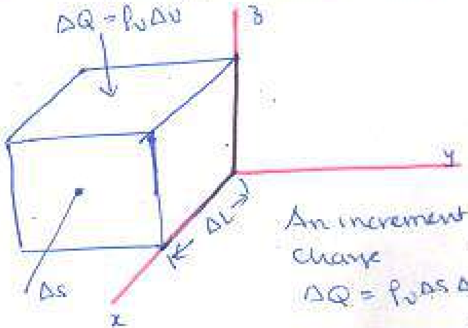
$$\boxed{\Delta I = \bar{J} . \Delta \bar{S}}$$

Thus total current flowing through a surface S is

$$\boxed{I = \int_S \bar{J} . ds}$$

Relationship between J and ρ_v

Current density may be related to the velocity of volume charge density at a point.



Consider the element of charge

$$\Delta Q = \rho_v \Delta v = \rho_v \Delta S \Delta L$$

Assume that the charge component possesses only an ~~x~~ component of velocity

In the time interval Δt, the element of charge has moved a distance Δx

∴ we have moved the charge

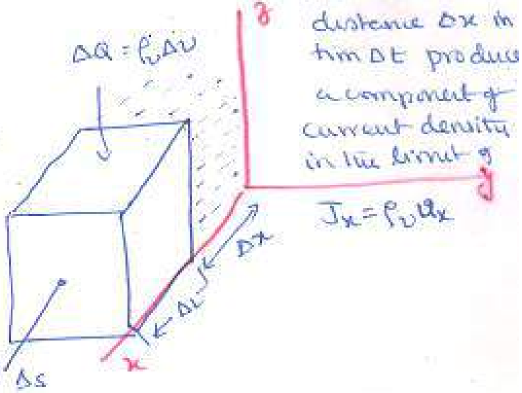
$$\Delta Q = \rho_v \Delta S \Delta x \text{ in time increment } \Delta t$$

$$\therefore \Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \Delta S \frac{\Delta x}{\Delta t}$$

As we take the limit w.r.t, we have

$$\Delta I = \rho_v \Delta S u_x$$

where u_x — ~~tot~~ x component of velocity



An increment of charge $\Delta Q = \rho_v \Delta S \Delta L$ which moves a distance Δx in time Δt produces a component of current density in the limit of

$$J_x = \rho_v u_x$$

The above equation in terms of current density

$$J_x = \rho_v u_x$$

Hence, in general

$$\boxed{\bar{J} = \rho_v \bar{u}} \longrightarrow \text{Convection Current density}$$

$\bar{I}$ — Convection Current

Convection Current, is distinct from Conduction current, does not involve conductors and consequently does not satisfy Ohm's law. It occurs when the current flow through an insulating medium such as liquid, etc.

Depending on how I is produced there are different kinds of current density.

① Convection Current density

② Conduction Current Density

③ ~~Displacement~~ Diffusion Current Density

Point Form of Ohm's law

Conduction current requires a conductor. When an electric field $\bar{E}$ is applied, the force on electron with charge $-e$ is

$$F = -eE$$

Since the electron is not in free space, it will not experience an average acceleration under the influence of the electric field. Rather, it suffers an ~~average acceleratn~~ constant collisions with the atomic lattice and drift from one atom to another.

If an electron ~~with~~ with mass 'm' is moving in an electric field $\bar{E}$ with an average drift velocity $\bar{u}$, according to Newton's law, the average momentum of the free electron must match the applied force:

Thus

$$\frac{m\bar{u}}{\tau} = -e\bar{E}$$

or $$\bar{u} = -\frac{e\tau}{m}\bar{E}$$ where τ - average time intervals between collisions.

If there are n electrons per unit volume, the electron charge density is given by $\rho_v = -ne$

Thus the conduction current density is

$$\bar{J} = \rho_v\bar{u} = \frac{ne^2\tau}{m}\bar{E} = \sigma\bar{E}$$

or $$\boxed{\bar{J} = \sigma\bar{E}}$$ where $\sigma - ne^2\tau/m$ is the conductivity of the conductor.

↑ point form of Ohm's law

CONTINUITY EQUATON (or CONTINUITY OF CURRENT)

The continuity equation of the current is based on the principle of conservation of charge. (The charge can neither be created not be destroyed)

When we consider any region bounded by a closed surface, the current through the closed surface is

$$\boxed{I = \oint_s \bar{J}\cdot d\bar{s}}$$

The current flows outwards from a closed surface. & Current means flow of positive charges.

According to principle of conservation of charge, there must be a decrea of an equal amount of positive charge inside the closed surface. Hence th outward rate of flow of positive charge gets balanced by the rate of decrease of charge inside the closed surface.

Let Q_i = Charge within the closed surface

$-\frac{dQ_i}{dt}$ - Rate of decrease of charge inside the closed surface

This rate of decrease must be same as rate of outward flow of charge, which is a current

$$\boxed{I = \oint_s \bar{J}\cdot d\bar{s} = -\frac{dQ_i}{dt}}$$

Integral form of Continuity Equation.

-ve sign indicates outward flow

Using the divergence theorem, change the surface integral into volume integra

Using the divergence theorem, change the surface integral into volume integra

$$\therefore \oint_S \bar{J}\cdot d\bar{s} = \int_{vol} (\nabla\cdot\bar{J})\,dv$$

$$\therefore -\frac{dQ_i}{dt} = \int_{vol} (\nabla\cdot\bar{J})\,dV$$

But $Q_i = \int_{vol} \rho_v\,dv$

ρ_v – volume charge densi

$$\therefore \int_{vol} (\nabla\cdot\bar{J})\,dV = -\frac{d}{dt}\left[\int_{vol} \rho_v\,dv\right] = -\int_{vol} \frac{\partial\rho_v}{\partial t}\,dv$$

For a constant surface, the derivatives becomes the partial derivatives

$$\therefore \int_{vol} \nabla\cdot\bar{J}\,dV = \int_{vol} -\frac{\partial v}{\partial t}\,dv$$

If the relation is true for any volume, it must be true even for incremental volume ΔV.

$$\therefore (\nabla\cdot\bar{J})\Delta V = -\frac{\partial\rho_v}{\partial t}\Delta v \quad \text{(or)} \quad \boxed{\nabla\cdot J = -\frac{\partial\rho_v}{\partial t}}$$

Point form or differential form of the continuity eq of current

Steady Current $\partial\rho_v/\partial t = 0$

$$\therefore \boxed{\nabla\cdot J = 0}$$

Relaxation Time

Consider a conducting material which is linear and homogeneous. The current density for such a material is,

$\bar{J} = \sigma\bar{E}$ where σ = conductivity

But $\bar{D} = \epsilon\bar{E}$ (for linear material)

$$\bar{E} = \frac{\bar{D}}{\epsilon}$$

$$\therefore \bar{J} = \sigma\frac{\bar{D}}{\epsilon} = \frac{\sigma}{\epsilon}\bar{D}$$

And point form of continuity equation states that

$$\nabla\cdot\bar{J} = \frac{-\partial\rho_v}{\partial t}$$

$$\therefore \nabla\cdot\left(\frac{\sigma}{\epsilon}\bar{D}\right) = -\frac{\partial\rho_v}{\partial t}$$

$$\frac{\sigma}{\epsilon}\nabla\cdot\bar{D} = \frac{-\partial\rho_v}{\partial t}$$

But $\nabla\cdot\bar{D} = \rho_v$

$$\therefore \frac{\sigma}{\epsilon}\rho_v = -\frac{\partial\rho_v}{\partial t}$$

$$\therefore \frac{\partial\rho_v}{\partial t} + \frac{\sigma}{\epsilon}\rho_v = 0$$

Resistance $R = \frac{V}{I} = \frac{\int \bar{E}\cdot d\bar{l}}{\oint \sigma\bar{E}\cdot d\bar{s}}$

Capacitance $C = \frac{Q}{V} = \frac{\epsilon\oint \bar{E}\cdot d\bar{s}}{\int \bar{E}\cdot d\bar{l}}$

Product of these two.

$$RC = \frac{\epsilon}{\sigma} = \tau_r$$

is the relaxation time of medium separating the conductors.

This is a differential equation in ρ_v whose solution is given by

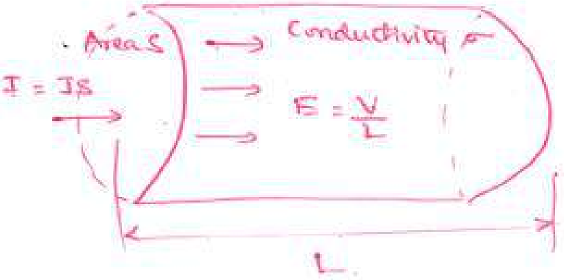
$$\boxed{\rho_v = \rho_0 e^{-(\sigma/\epsilon)t} = \rho_0 e^{-t/\tau}}$$

ρ_0 – charge density at $t=0$.

This shows that if there is a temporary imbalance of electrons inside the given material, the charge density decays exponentially with a time constant $\tau = \epsilon/\sigma$ sec. This is called relaxation tim

RESISTANCE OF A CONDUCTOR (OHM'S LAW.

Consider the Voltage V is applied to a conductor of length L having uniform cross-section S, as shown.



The direction of $\bar{E}$ is same as the direction of conventional current, which is opposite to the flow of electrons. The electric field applied is uniform and its magnitude is given by

$$E = \frac{V}{L}$$

The conductor has uniform cross section S and hence we can write

$$I = \int_S \bar{J}.d\bar{S} = JS$$

The current direction is normal to the surface. $(\cos\theta = 1)$

Thus $J = \frac{I}{S} = \sigma \bar{E}$

$$= \frac{\sigma V}{L}$$ where σ is the conductivity of the material.

$$V = \frac{JL}{\sigma} = \frac{IL}{\sigma S} = \left(\frac{L}{\sigma S}\right) I$$

$$\boxed{\therefore R = \frac{V}{I} = \frac{L}{\sigma S}}$$ Is known as Ohm's Law when the field is uniform

When the field is non-uniform, the general expression for resistance is

$$\boxed{R = \frac{V_{AB}}{I} = \frac{-\int_B^A \bar{E}.d\bar{L}}{\int_S \bar{J}.d\bar{S}} = \frac{-\int_B^A \bar{E}.d\bar{L}}{\int_S \sigma \bar{E}.d\bar{S}}}$$

= Line integral taken between two equipotential surfaces in the conductor / Surface integral giving more current flowing through the material.

Power is dissipaped through the conductor due to resistance. This is expressed from Jule's law as $P = \int \bar{E}.\bar{J} dv$ watts

For the conductor having uniform cross section

$$P = \int_v \bar{E}.\bar{J}.(d\bar{l}\, ds) = \int_l \bar{E}.d\bar{l} \int_S \bar{J}.d\bar{S} = VI = J^2 R \text{ watts}.$$

Given $z<0$ is a region of a linear dielectric of relative permittivity 6.5 and $z>0$ is free space. Electric Field in the free space region is $(-3\bar{a}_x+4\bar{a}_y-2\bar{a}_z)$ V/m. Find (i) $\bar{D}$ for $z>0$; (ii) tangential components of $\bar{D}$ and $\bar{E}$ on the boundary of $z<0$ region. (Feb 2005)

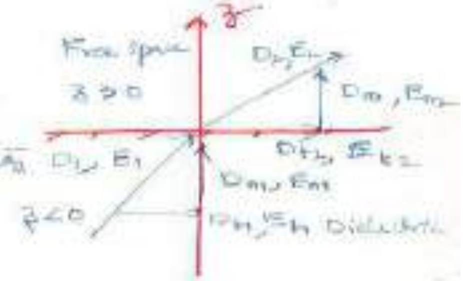
Let the two regions be region 1 for $z<0$ — dielectric region
2 for $z>0$ — free space

Then $\epsilon_{r1}=6.5$, $\epsilon_{r2}=1$ and $\bar{E}_2=-3\bar{a}_x+4\bar{a}_y-2\bar{a}_z$

(i) D for $z>0$ is $\bar{D}_2$

$$\bar{D}_2=\epsilon_2\bar{E}_2=8.854\times10^{-12}(-3\bar{a}_x+4\bar{a}_y-2\bar{a}_z)$$
$$=(-26.56\bar{a}_x+35.41\bar{a}_y-17.7\bar{a}_z)\times10^{-12}\ \text{c/m}^2$$

(ii) From the figure.



$$\bar{E}_2=\bar{E}_{t2}+E_{N2}$$
$$\therefore E_{N2}=-2\bar{a}_z$$
$$\bar{E}_{t2}=\bar{E}_2-E_{N2}=-3\bar{a}_x+4\bar{a}_y-2\bar{a}_z+2\bar{a}_z$$
$$=-3\bar{a}_x+4\bar{a}_y$$

At the boundary tangential components are continuous

$$\bar{E}_{t1}=\bar{E}_{t2}=-3\bar{a}_x+4\bar{a}_y$$
$$|E_{t1}|=\sqrt{3^2+4^2}=5\text{V/m}$$

flux density $D_{t1}=\epsilon E_{t2}=\epsilon(-3\bar{a}_x+4\bar{a}_y)$
$$=6.5\times8.854\times10^{-12}(-3\bar{a}_x+4\bar{a}_y)$$
$$=(-17.2\bar{a}_x+23.02\bar{a}_y)\times10^{-11}\ \text{c/m}^2$$

$$|\bar{D}_{t1}|=0.287\ \text{nC/m}^2.$$

The construction of a paper capacitor is as follows. Aluminium foil of 100 cm² area is placed on both sides of paper of thickness 0.03 mm. If the dielectric constant of paper is given by as 3 and dielectric breakdown strength is 200 kV/cm, what is the rating of the capacitor? (Nov 2005, 2004)

Capacitance $C=\dfrac{\epsilon S}{d}$

d – paper thickness – 0.03 mm. $=0.03\times10^{-3}$ m
S – area $=100\text{cm}^2=100\times10^{-4}=10^{-2}\text{m}^2$
$\epsilon_r=3$
$E_{max}=200\ \text{kV/cm}=200\times10^3\times10^2\ \text{V/m}=2\times10^7\ \text{V/m}$

$$\therefore \text{capacitance } C=\frac{\epsilon S}{d}=\frac{\overset{\epsilon_0}{8.854\times10^{-12}}\times\overset{\epsilon_r}{3}\times10^{-2}}{3\times10^{-5}}$$
$$=8.854\ \text{nF}$$

maximum operating voltage $V=E.d=2\times10^7\times3\times10^{-5}=600\text{V}$
Maximum energy stored $W_E=\frac{1}{2}CV^2=\frac{1}{2}\times8.854\times10^{-9}(600)^2=1.6\text{m}$
The rating of the capacitor is $V_{max}=600\text{V}$
and $W_{Emax}=1.6\text{mJ}$

Questions

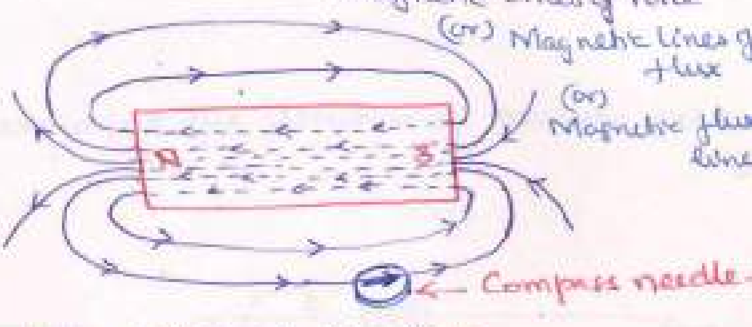
1. Describe the dielectric material in brief (May2008)
2. State and prove the conditions at the boundary between two dielectrics (May 2008, 2011)
3. Explain polarization of dielectric materials (May2008, 09, Nov 2010
4. Describe electric displacement vector in dielectrics (May2008, 10
5. Derive the expression for the capacitance of a parallel plate capacitor with two dielectrics (May2010)
6. State & explain continuity equation of current in integral form and point form (May 2010, Nov 2011)
   May 2011
7. Derive Ohm's law in point form. (May2010, 11)
8. Explain and derive the boundary conditions for a conductor and free space interface (May2011, 12)
9. Derive the expression for the energy stored in a parallel plate capacitor (May 2011)
10. State & ~~explain~~ derive boundary conditions between two perfect dielectrics (Nov 2011, May 2012)
11. Derive the expression for leakage resistance of a coaxial cable (May2010)

UNIT IV : MAGNETOSTATICS

Magnetic Field & its Properties :

The region around a magnet within which the influence of the magnet can be experienced is called magnetic field.

Such a field is represented by imaginary lines around the magnet which are called magnetic lines of force. Introduced by Michael Faraday.



The direction of such lines is always from N pole to S pole external to the magnet.

Difference between Electric Flux Lines & Magnetic Flux lines:

- Electric flux lines originate from an isolated positive charge and diverge to terminate at infinity.
- While for negative charge, electric flux lines converge on a charge starting from infinity.
- But in case of magnetic flux, the lines of flux exists in pairs of poles only. i.e, An Isolated magnetic pole cannot exist. Hence, every magnetic flux line starting from north pole must end at south pole and complete the path from south to north internal to the magnet.

  Thus, magnetic flux lines exist in the form of closed loop.

This is true whether the field is due to permanent magnet or due to conductor carrying direct current.

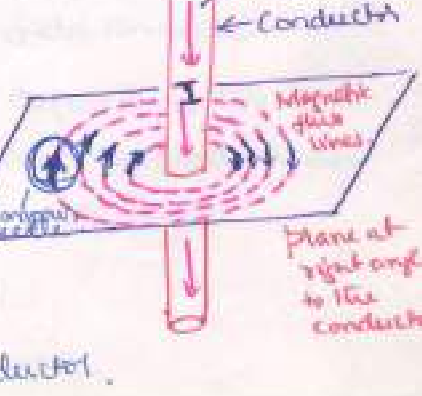
In 1820
Hans Christian Oersted's Experiment
to find the relationship between electricity & magnetism

Magnetic Field due to Current Carrying Conductor.

When a straight conductor carries a direct current, it produces a magnetic field around it, all along its length.

The lines of force in such a case are of concentric circles in the planes at right angles to the conductor.

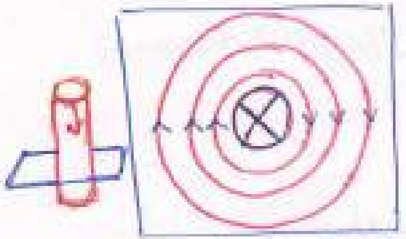
The direction of these circles depends on the direction of current passing through the conductor.

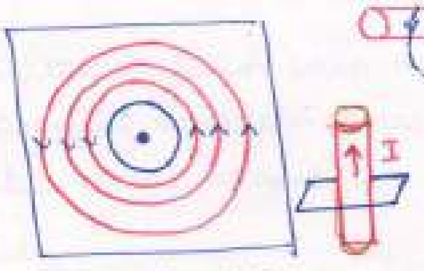


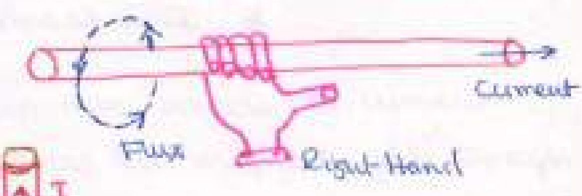
As long as direction of current is constant and current is time independent, the magnetic lines of force are also constant, static and time independent, giving a steady magnetic field in the space around the conductor.

Direction of Magnetic Field.

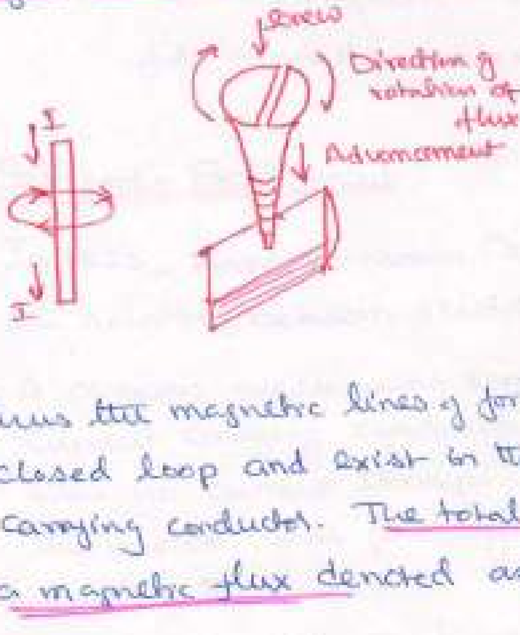
Right Hand Thumb Rule: To determine the direction of magnetic field

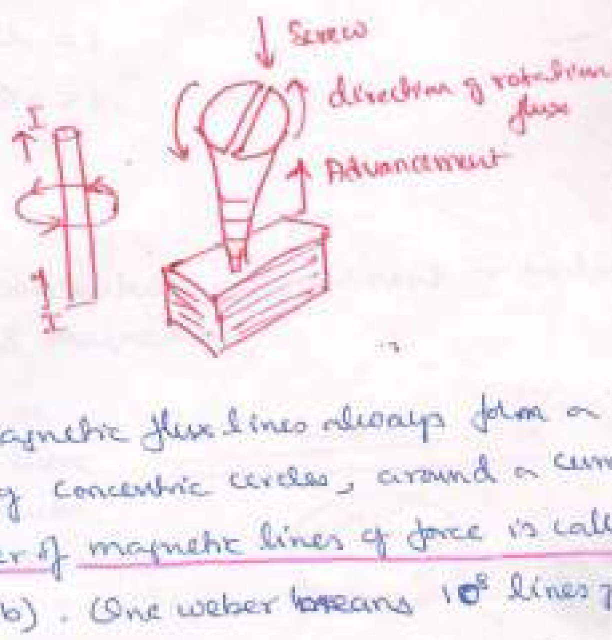

Current going into the plane


Current ~~goin~~ coming out of the plane



Right Handed Screw Rule





Thus the magnetic lines of force or magnetic flux lines always form a closed loop and exist in the form of concentric circles, around a current carrying conductor. The total number of magnetic lines of force is called a magnetic flux denoted as ϕ (wb). One weber means 10^8 lines of force

Magnetic Field Intensity

The quantitative measure of strongness or weakness of the magnetic field is given by magnetic field intensity or magnetic field strength.

Def:- MFI at any point in the magnetic field is defined as the force experienced by a unit north pole of one weber strength, when placed at that point. $\overline{H}$ (N/wb or, A/m or AT/m)

Is similar to EFI ($\overline{E}$) in electrostatics.

Magnetic Flux Density

Total magnetic flux (total magnetic lines of force) crossing a unit area in a plane at right angles to the direction of flux is called magnetic flux density $\bar{B}$ (Wb/m² or Tesla)

Similar to Electric Flux Density $\bar{D}$ in electrostatics.

Relation between $\bar{B}$ & $\bar{H}$

In electrostatics $\bar{E} = \frac{\bar{D}}{\epsilon}$ (or) $\epsilon\bar{E} = \bar{D}$ ϵ – is the permittivity of the region

In magnetostatics $\bar{B} = \mu\bar{H}$ μ – permeability.

permeability (μ) – Is the ability or ease with which the current carrying conductor forces the magnetic flux through the region around it.

$\mu = \mu_0\mu_r$

μ_0 – permeability of free space $= 4\pi \times 10^{-7}$ H/m.

μ_r – relative permeability

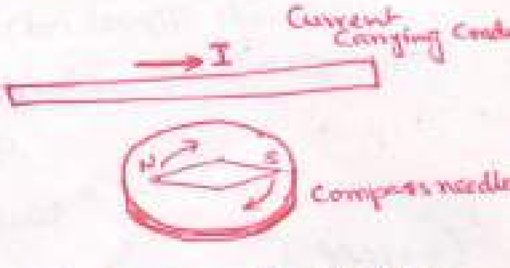
For all non magnetic media $\mu_r = 1$

for magnetic materials $\mu_r > 1$.

Oersted's Experiment:

In 1820, Hans Christian Oersted conducted an experiment to find out the relation between electricity and magnetism.

A compass needle was kept under a current carrying conductor. Where there was no current through the conductor, the needle was pointing along north & south of the earth.



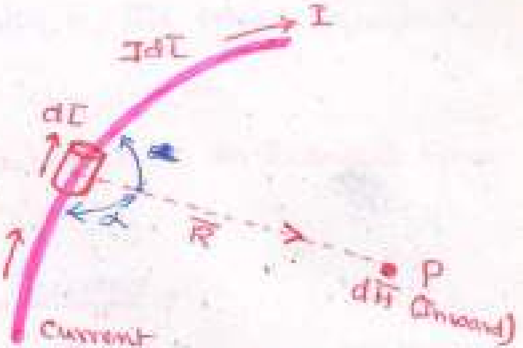
But when the conductor carries current then the needle tend to move in right angle to the conductor.

From this experiment, Oersted showed that an electric current produces a magnetic field.

This experiment proved to be one of the stepping stones towards Maxwell's equations.

BIOT - SAVART'S LAW

Consider a conductor carrying a direct current I and a steady magnetic field produced around it. The Biot-Savart's law allows us to obtain the differential magnetic field intensity $d\bar{H}$ produced at point P, due to differential current element $I d\bar{L}$



Statement: The differential magnetic field intensity $d\bar{H}$ produced at a point P by the differential element $Id\bar{L}$ is

- proportional to the product of $Id\bar{L}$ & the angle α between the line joining P to the element and the element, and
- Inversely proportional to the square of the distance R between P & the element.

That is $$dH \propto \frac{I\,dL\,\sin\alpha}{R^2}$$

or $$dH = \frac{K I\,dL\,\sin\alpha}{R^2}$$

Where K is the constant of proportionality in SI Units $K = \frac{1}{4\pi}$, so

$$dH = \frac{1}{4\pi}\frac{I\,dL\,\sin\alpha}{R^2} \longrightarrow ①$$

Let us express this equation in vector form

Let $d\bar{L}$ = the magnitude of vector length $d\bar{L}$ and

$\bar{a}_R$ = Unit vector in the direction from differential current element to P.

From the rule of the cross product

$$d\bar{L} \times \bar{a}_R = dL\,|a_R|\,\sin\alpha = dL\,\sin\alpha \quad (\because |a_R| = 1) \longrightarrow ②$$

Now replacing eq ① with eq ②

$$d\bar{H} = \frac{I\,d\bar{L} \times \bar{a}_R}{4\pi R^2} \text{ A/m}$$

But $\bar{a}_R = \frac{\bar{R}}{|R|} = \frac{\bar{R}}{R}$

Hence $$\boxed{d\bar{H} = \frac{I\,d\bar{L} \times \bar{a}_R}{4\pi R^2} = \frac{I\,d\bar{L} \times \bar{R}}{4\pi R^3} \text{ A/m}}$$

Mathematical form of Biot-Savart's law

All such differential elements constitute the entire conductor. Hence to obtain total magnetic field intensity $\bar{H}$, the above equation takes an integral form as,

$$\boxed{\bar{H} = \oint_L \frac{I\,d\bar{L} \times \bar{a}_R}{4\pi R^2}}$$ — Biot-Savart's Law in Integral Form.

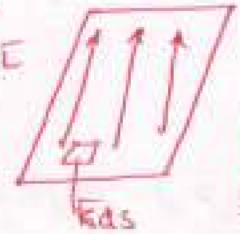
Biot-Savart's Law Interms of Distributed Sources.

Just as we can have different charge distributions, we can have different current distributions: line current, surface current & volume current.

If we define $\bar{K}$ – Surface current density A/m
$\bar{J}$ – volume current density A/m²

The source elements are related as

$$I\,d\bar{L} = \bar{K}\,ds = \bar{J}\,dv$$



Thus interms of distributed current sources, the Biot-Savart's Law becomes

$$\bar{H} = \int_L \frac{I\,d\bar{L} \times \bar{a}_R}{4\pi R^2} \quad \text{(line current)}$$

$$\bar{H} = \int_S \frac{\bar{K}\,ds \times \bar{a}_R}{4\pi R^2} \quad \text{(Surface current)}$$

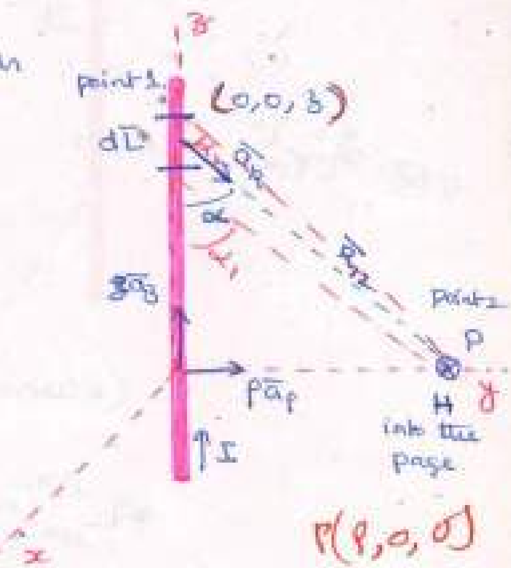
$$\bar{H} = \int_V \frac{\bar{J}\,dv \times \bar{a}_R}{4\pi R^2} \quad \text{(volume current)}$$

where $\bar{a}_R$ is a unit vector pointing from the differential element of current to the point of interest.

Applications of Biot-Savart's Law

1. $\overline{H}$ due to Infinitely Long Straight Conductor:

Consider an infinitely long straight conductor, along z-axis. To find $\overline{H}$ at a point P which is at a distance r from z axis, consider a small differential element at point 1 along the z-axis, at a distance z from the origin



$$\therefore I\,d\overline{L} = I\,dz\,\overline{a}_z$$

Now the displacement (distance) vector joining point-1 to point-2 is $\overline{R}_{12}$ and can be written as $(0-z)\overline{a}_z+(\rho-0)\overline{a}_\rho$

$$\overline{R}_{12} = -z\overline{a}_z + \rho\overline{a}_\rho$$

$$\overline{a}_{R_{12}} = \frac{\overline{R}_{12}}{|R_{12}|} = \frac{\rho\overline{a}_\rho - z\overline{a}_z}{\sqrt{\rho^2+z^2}}$$

$$\therefore d\overline{L}\times\overline{a}_{R_{12}} = \frac{\rho\,dz\,\overline{a}_\phi}{\sqrt{\rho^2+z^2}} \qquad (\because d\overline{L} = dz\,\overline{a}_z \text{ \& } \overline{R}_{12} = \rho\overline{a}_\rho - z\overline{a}_z$$

$$a_{R_{12}} = \frac{R_{12}}{|R_{12}|} = \frac{\rho\overline{a}_\rho - z\overline{a}_z}{\sqrt{\rho^2+z^2}}$$

$$(\because I\,d\overline{L}\times\overline{a}_{R_{12}} = \frac{I\rho\,dz\,\overline{a}_\phi}{\sqrt{\rho^2+z^2}}$$

$$d\overline{L}\times\overline{a}_{R_{12}} \qquad |a_{R_{12}}| = 1$$

$$= |d\overline{L}|\,|a_{R_{12}}|\sin\alpha \text{ (or)}$$

$$= (dz\,\overline{a}_z)\times\frac{(\rho\overline{a}_\rho - z\overline{a}_z)}{\sqrt{\rho^2+z^2}}$$

$$= \frac{\rho\,dz\,\overline{a}_z\times\overline{a}_\rho - dz\,z\,\overline{a}_z\times\overline{a}_z}{\sqrt{\rho^2+z^2}}$$

$$= \frac{\rho\,dz\,\overline{a}_\phi - 0}{\sqrt{\rho^2+z^2}}$$

$$(\because \overline{a}_z\times\overline{a}_\rho = \overline{a}_\phi$$
$$\overline{a}_z\times\overline{a}_z = 0)$$

$$\therefore dH = \frac{I\,d\overline{L}\times\overline{a}_{R_{12}}}{4\pi R_{12}^2} = \frac{I\rho\,dz\,\overline{a}_\phi}{4\pi(\rho^2+z^2)^{3/2}}$$

$$\therefore \overline{H} = \int_{z=-\infty}^{\infty}\frac{I\rho\,dz\,\overline{a}_\phi}{4\pi(\rho^2+z^2)^{3/2}}$$

Letting $z = \rho\cot\alpha$, $dz = -\rho\,\mathrm{cosec}^2\alpha\,d\alpha$

$z^2 = \rho^2\cot^2\alpha$ and $z = -\infty$ $\alpha = \pi = \alpha_1$

$z = +\infty$ $\alpha = \pi/2 = \alpha_2 = 0$

$z = 0$ to ∞
$\alpha_1 = 90°$ $\alpha_2 = 0°$

$$\therefore \overline{H} = \frac{I}{4\pi}\int_{\alpha_1}^{\alpha_2}\frac{\rho(-\rho\,\mathrm{cosec}^2\alpha)\,d\alpha}{(\rho^2+\rho^2\cot^2\alpha)^{3/2}}\overline{a}_\phi$$

$$= -\frac{I}{4\pi}\int_{\alpha_1}^{\alpha_2}\frac{\rho^2\,\mathrm{cosec}^2\alpha\,d\alpha}{\rho^3(1+\cot^2\alpha)^{3/2}} = -\frac{I}{4\pi\rho}\int_{\alpha_1}^{\alpha_2}\frac{\mathrm{cosec}^2\alpha\,d\alpha}{\mathrm{cosec}^3\alpha}\overline{a}_\phi$$

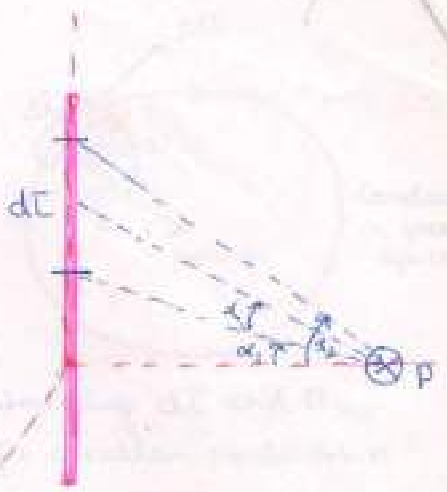
$$\boxed{\overline{H} = \frac{-I}{4\pi\rho}\int_{\alpha_1}^{\alpha_2}\sin\alpha\,d\alpha\,\overline{a}_\phi = \frac{1}{4\pi\rho}\left[\cos\alpha_2 - \cos\alpha_1\right]\overline{a}_\phi}$$

$\overline{H}$ is always along the unit vector $\overline{a}_\phi$ (ie along concentric circular paths) irrespective of the length of the wire

cases

When $\alpha_1 = 90°$ & $\alpha_2 = 0°$ (0 to ∞) $\overline{H} = \frac{I}{4\pi\rho}\overline{a}_\phi$	When $\alpha_1 = 180°$ $\alpha_2 = 0°$ (−∞ to ∞) $\overline{H} = \frac{1}{2\pi\rho}\overline{a}_\rho$

$$\overline{H} = \int_{-\infty}^{\infty} dH = \int_{-\infty}^{\infty} \frac{I\rho\, dz\, \overline{a_\phi}}{4\pi(\rho^2+z^2)^{3/2}}$$



Use $z = \rho\tan\alpha \quad z^2 = \rho^2\tan^2\alpha$

$dz = \rho\sec^2\alpha\, d\alpha$

for $z = \infty \quad \alpha_1 = +\pi/2$

$z = -\infty \quad \alpha_2 = -\pi/2$

$$\therefore H = \int_{\alpha_1 = \pi/2}^{\alpha_2 = \pi/2} \frac{I\rho\cdot\rho\sec^2\alpha\, d\alpha\, \overline{a_\phi}}{4\pi(\rho^2+\rho^2\tan^2\alpha)^{3/2}}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{I\rho^2\sec^2\alpha\, d\alpha\, \overline{a_\phi}}{4\pi\rho^3\sec^3\alpha} \quad (\because 1+\tan^2\alpha = \sec^2\alpha)$$

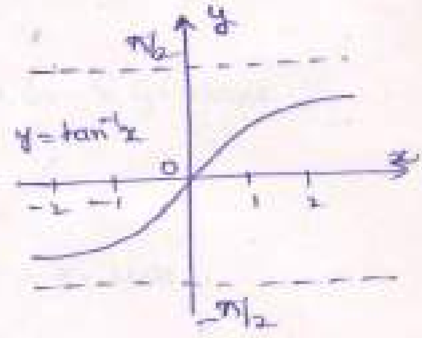
$$= \frac{I}{4\pi\rho}\int_{-\pi/2}^{\pi/2} \cos\alpha\, d\alpha\, \overline{a_\phi} = \frac{I}{4\pi\rho}\left[\sin\alpha\right]_{-\pi/2}^{\pi/2} \overline{a_\phi}$$

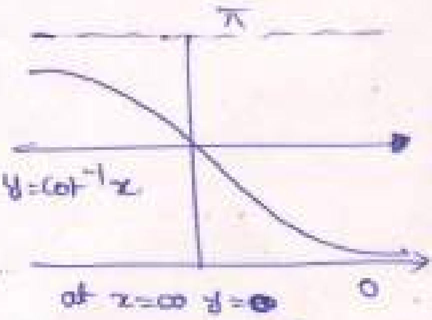
$$= \frac{I}{4\pi\rho}\left[1-(-1)\right]\overline{a_\phi} = \frac{2I}{4\pi r}\overline{a_\phi}$$

$$\boxed{\therefore \overline{H} = \frac{I}{2\pi\rho}\overline{a_\phi} \quad A/m}$$

$$\boxed{\overline{B} = \mu\overline{H} = \frac{\mu I}{2\pi\rho}\overline{a_\phi} \quad wb/m^2}$$

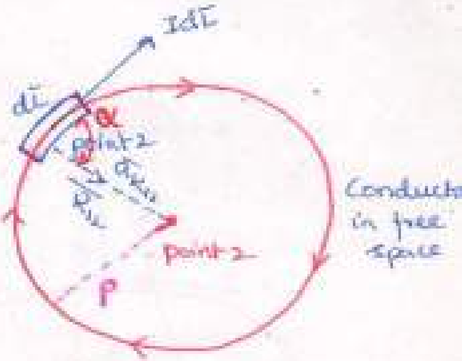
Inverse Function	Domain	Interval
$y = \sin^{-1}x$	$[-1, 1]$	$[-\frac{\pi}{2}, \pi/2]$
$y = \cos^{-1}x$	$[-1 \ 1]$	$[0, \pi]$
$y = \tan^{-1}x$	$[-\infty \ \infty]$	$(-\frac{\pi}{2}, \pi/2]$
$y = \cot^{-1}(x)$	$[-\infty \ \infty]$	$[0, \pi)$
$y = \sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi], y \neq \pi/2$
$y = \csc^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$(-\frac{\pi}{2}, \pi/2)\ y \neq 0$





at $x = \infty \ y = 0$
$-\infty \ y = \pi$

$\overline{H}$ at the Centre of a Circular Conductor.



Consider the current carrying conductor arranged in a circular form as shown in figure.

To find $\overline{H}$ at the centre of the circular loop, consider the differential length $\overline{dL}$ at point 1.

Thus

$$I\overline{dL} \times \overline{a}_{R_{12}} = I|\overline{dL}||\overline{a}_{R_{12}}|\sin\alpha\,\overline{a}_N$$

$\overline{a}_N$ — Unit vector normal to the plane containing $\overline{dL}$ and $\overline{a}_{R_{12}}$ ie normal to the plane in which the circular conductor is lying.

According to Biot - Savart's law

$$\therefore d\overline{H} = \frac{I\overline{dL} \times \overline{a}_{R_{12}}}{4\pi R_{12}^2} = \frac{I\,dL \sin\alpha\,\overline{a_N}}{4\pi R_{12}^2} \qquad (\because R = R_{12} = \text{Radius})$$

∴ Total magnetic field

$$\overline{H} = \oint d\overline{H} = \oint \frac{I\,dL\sin\alpha}{4\pi R_{12}^2}\,\overline{a}_N = \frac{I\sin\alpha\,\overline{a_N}}{4\pi R_{12}^2}\oint dL$$

But $\oint dL = 2\pi R_{12}$ — Circumference of the circle

$$\therefore \overline{H} = \frac{I\sin\alpha \cdot 2\pi R_{12}\,\overline{a}_N}{4\pi R_{12}^2} = \frac{I\sin\alpha}{2R_{12}}\,\overline{a}_N$$

As ~~the~~ $I\overline{dL}$ is tangential to the circle & R_{12}, angle α must be 90°

$$\therefore \overline{H} = \frac{I\sin 90^\circ}{2R_{12}}\,\overline{a}_N = \frac{I}{2R_{12}}\,\overline{a}_R \quad A/m$$

$\overline{a}_N = \overline{a}_3$ if the circular loop placed in x y - plane

$$\therefore \boxed{\overline{H} = \frac{I}{2R}\,\overline{a}_3} \; A/m$$

$$\& \quad \boxed{\overline{B} = \frac{\mu_0 I}{2R}\,\overline{a}_3} \; wb/m^2 \qquad \because B = \mu_0 H$$

If $R_{12} = \rho$

(or)

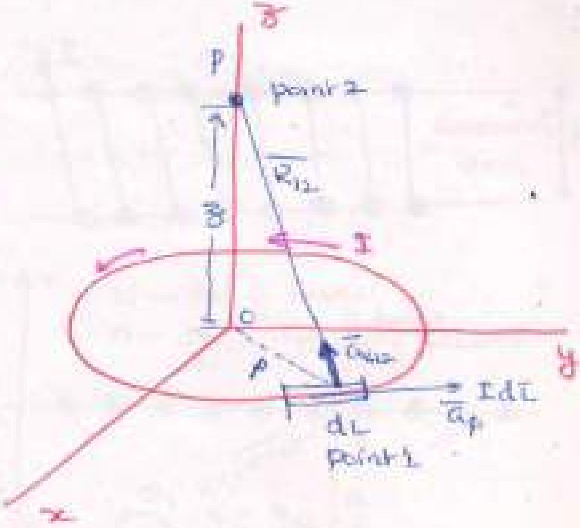
$$\boxed{\overline{H} = \frac{I}{2\rho}\,\overline{a}_3 \quad A/m}$$

$$\boxed{B = \frac{\mu_0 I}{2\rho}\,\overline{a}_3 \quad wb/m^2}$$

$\bar{H}$ on the axis of Circular loop

Consider a circular loop carrying a direct current I, placed in xy plane, with z-axis as its axis. Consider a point $\bar{P}$ at a distance z from the origin (plane) along the z axis.

Let the radius of the circular loop be ρ. Consider the differential length $d\bar{L}$ of the circular loop as shown in figure.



$$Id\bar{L} = I(d\rho\bar{a}_\rho + \rho d\phi\bar{a}_\phi + dz\bar{a}_z)$$

Since IdL is the tangential at point 1 in $\bar{a}_\phi$ direction. E, the loop is in xy plane

$$Id\bar{L} = I\rho d\phi\bar{a}_\phi$$

Now $$\bar{a}_{R_{12}} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{-\rho\bar{a}_\rho + z\bar{a}_z}{\sqrt{\rho^2+z^2}}$$

$$\therefore d\bar{L}\times\bar{a}_{R_{12}} = I\rho d\phi\bar{a}_\phi \times \left(\frac{-\rho\bar{a}_\rho + z\bar{a}_z}{\sqrt{\rho^2+z^2}}\right) = \frac{I[z\rho d\phi\bar{a}_\rho + \rho^2 d\phi\bar{a}_z]}{\sqrt{\rho^2+z^2}}$$

Now according to Biot-Savart's law

$\bar{a}_\phi\times\bar{a}_\rho = -\bar{a}_z$

$$d\bar{H} = \frac{Id\bar{L}\times\bar{a}_{R_{12}}}{4\pi R_{12}^2} = \frac{I[z\rho d\phi\bar{a}_\rho + \rho^2 d\phi\bar{a}_z]}{4\pi\sqrt{\rho^2+z^2}\left(\sqrt{\rho^2+z^2}\right)^2}$$

Note: It can be observed that though $d\bar{H}$ consists of two components $\bar{a}_\rho$ and $\bar{a}_z$, due to radial symmetry all $\bar{a}_\rho$ components are going to cancel each other. So $\bar{H}$ exists only along the axis in $\bar{a}_z$ direction

$$\therefore \bar{H} = \frac{I}{4\pi}\int_{\phi=0}^{2\pi}\frac{\rho^2 d\phi}{(\rho^2+z^2)^{3/2}}\bar{a}_z = \frac{I}{4\pi}\frac{\rho^2\bar{a}_z}{(\rho^2+z^2)^{3/2}}\int_{\phi=0}^{2\pi}d\phi$$

$$= \frac{I\rho^2\bar{a}_z}{4\pi(\rho^2+z^2)^{3/2}}[\phi]_0^{2\pi} = \frac{I\rho^2 2\pi\bar{a}_z}{4\pi(\rho^2+z^2)^{3/2}} =$$

$$\boxed{\bar{H} = \frac{I\rho^2}{2(\rho^2+z^2)^{3/2}}\bar{a}_z \quad A/m}$$

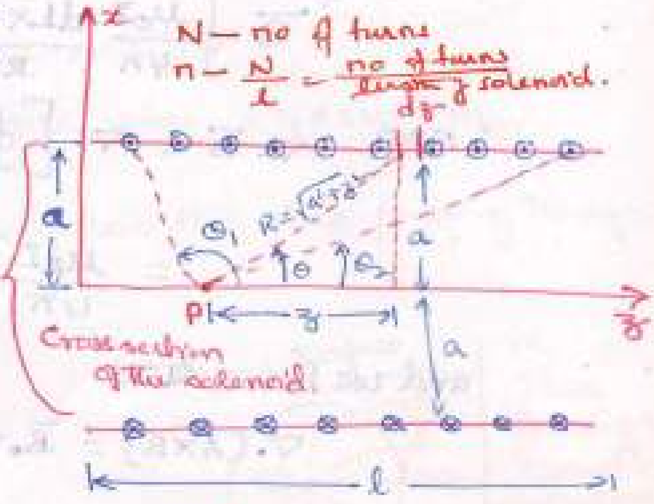
ρ – radius of the circular loop

z – distance of point P along the axis.

When z = 0 $$\boxed{\bar{H} = \frac{I\rho^2}{2(\rho^2)^{3/2}}\bar{a}_z = \frac{I}{2\rho}\bar{a}_z \quad A/m}$$

$\overline{H}$ on the axis of a Solenoid

A solenoid is a cylindrical shaped coil consisting of a large number of closely spaced turns of insulated wire wound usually on a non-magnetic frame. Such an arrangement may be used in electromagnetic relays and some of the electrical instruments.



Consider a solenoid of length 'l' and radius 'a' consists of 'N' turns of wire carrying current 'I'. Now let us consider the cross section of the solenoid as shown in figure. Since the solenoid consists of circular loops,

$$dH_z = \frac{I\,dL\,\sin\theta}{4\pi R^2}$$

from the eq (previous)

$$\overline{H} = \frac{I\rho^2}{2(\rho^2+z^2)^{3/2}}\overline{a_z}$$

$$\text{(or)}\ \overline{dH} = \frac{I\,dL\,\rho^2}{2(\rho^2+z^2)^{3/2}}\overline{a_3}$$

Contribution of to the magnetic field H at P by an element of the solenoid length dz is

$$= \frac{I\,dL\,a^2}{2(a^2+z^2)^{3/2}}$$

$$= \frac{Ia^2 n\,dz}{2[a^2+z^2]^{3/2}} \quad (\because dL = n\,dz = \tfrac{N}{l}dz)$$

From the figure $\tan\theta = a/z$ that is

$\Rightarrow z = a\cot\theta$

$$dz = -a\,\mathrm{cosec}^2\theta\,d\theta \quad = \frac{-(z^2+a^2)^{3/2}}{\cancel{a^2}\ \sin\theta\,d\theta}$$

Hence,

$$dH_z = -\frac{nI}{2}\sin\theta\,d\theta$$

$$H_z = -\frac{nI}{2}\int_{\theta_1}^{\theta_2}\sin\theta\,d\theta\,\overline{a_z}$$

$$\overline{H}_z = \frac{-nI}{2}\left[-\cos\theta\right]_{\theta_1}^{\theta_2}\overline{a_3}$$

$$\frac{-nI}{2}\left[-\cos\theta_1+\cos\theta_2\right]$$

$$= \frac{nI}{2}\left[\cos\theta_1-\cos\theta_1\right]$$

Thus

$$\overline{H} = \frac{nI}{2}\left[\cos\theta_2-\cos\theta_1\right]\overline{a_z}$$

Substituting $n = \frac{N}{l}$ gives

$$\boxed{\overline{H} = \frac{NI}{2l}\left[\cos\theta_2-\cos\theta_1\right]\overline{a_3}}$$

At the center of the solenoid

$$\cos\theta_2 = \frac{l/2}{(a^2+l^2/4)^{1/2}} = -\cos\theta_1$$

$$\boxed{\therefore \overline{H} = \frac{Inl}{2[a^2+l^2/4]^{1/2}}\overline{a_3}}$$

If $l >> a$ or $\theta_2 \simeq 0°$, $\theta_1 \simeq 180°$

$$\boxed{H = nI\overline{a_z} = \frac{NI}{l}\overline{a_3}}$$

$$dH = \frac{Ia^2n}{2[a^2+z^2]^{3/2}}\left[-a\,\mathrm{cosec}^2\theta\,d\theta\right]$$

$$dH = \frac{-Ia^3n\,\mathrm{cosec}^2\theta\,d\theta}{2[a^2+a\cot^2\theta]^{3/2}}$$

$$= \frac{-Ia^3n\,\mathrm{cosec}^2\theta\,d\theta}{2a^3[1+\cot^2\theta]}$$

$$= \frac{-I\cancel{a^3}n\,\mathrm{cosec}^2\theta\,d\theta}{2\cancel{a^3}\,\mathrm{cosec}^3\theta}$$

$$= \frac{-In}{2}\sin\theta\,d\theta.$$

Magnetic Flux Density - Maxwells' Equation

The magnetic flux density $\bar{B}$ is similar to the electric flux density $\bar{D}$. As $\bar{D} = \epsilon_0 \bar{E}$ in free space, the magnetic flux density $\bar{B}$ is related to the magnetic field Intensity $\bar{H}$ according to

$$\boxed{\bar{B} = \mu_0 \bar{H}}$$

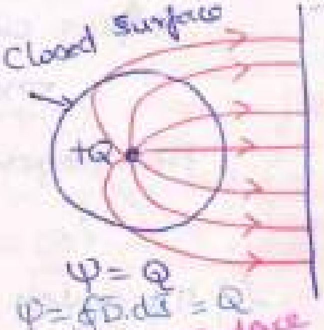
where $\bar{B}$ is measured in Wb/m^2 or Tesla

μ_0 - permeability of free space $= 4\pi \times 10^{-7}$ H/m

Now let us define magnetic flux ϕ as the flux passing through any designated area

$$\boxed{\phi = \int_s \bar{B}.d\bar{s}}$$

where ϕ is in webers



Flux leaving closed surface due to Isolated charg

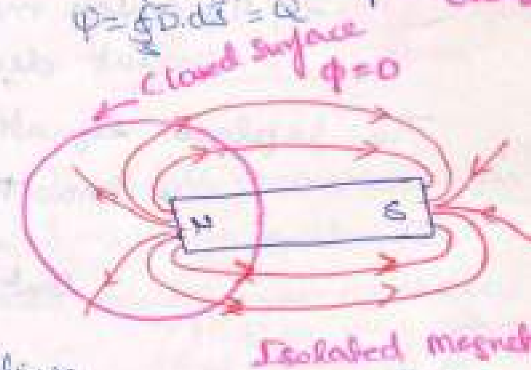
$\psi = Q$

$\psi = \oint_s \bar{D}.d\bar{s} = Q$

In electrostatics the total flux passing through any closed surface is equal to the charge enclosed (Gauss' law)

$$\psi = \oint_s \bar{D}.d\bar{s} = Q$$

The charge Q is the source of lines of electric flux and these lines begin on +ve charge and terminated on -ve charge



Isolated magnetic charge.

$\phi = \oint_s \bar{B}.d\bar{s} = 0$

Unlike electric flux lines, the magnetic flux lines always close upon themselves. This is because it is not possible to have an isolated magnetic poles or magnetic charges.

Maxwell's Eq from Gauss' law

Thus the total flux through a closed surface in a magnetic field must be zero, that is

$\boxed{\oint \bar{B}.d\bar{s} = 0} \rightarrow$ Law of Conservation of Magnetic flux (or) Gauss' law of magnetostatics.

Now applying divergence theorem

$\oint \bar{B}.d\bar{s} = \int_v \nabla \cdot \bar{B} dv = 0$

or $\nabla \cdot \bar{B} \int dv = 0$

$\Rightarrow \boxed{\nabla \cdot \bar{B} = 0}$ 4th Maxwells Eq.

The Four Maxwells Equations applied to Static Electric Fields & Steady Magnetic Field

	point form	Integral form
Gauss' law	$\nabla \cdot \bar{D} = \rho_v$	$\oint_s \bar{D}.d\bar{s} =$
	$\nabla \times \bar{E} = 0$	$\oint \bar{E}.d\bar{l} =$
Ampere's law	$\nabla \times \bar{H} = \bar{J}$	$\int \bar{H}.d\bar{l} = \int$
	$\nabla \cdot \bar{B} = 0$	$\int \bar{B}.d\bar{s}$

Maxwell's Second Equation for Magnetic Flux Density.

From Biot - Savart law.

The magnetic flux density at distance R from a current carrying conductor is

$$\overline{B} = \oint \frac{\mu_0 I}{4\pi} \frac{d\overline{L}\times\overline{a_R}}{R^2}$$

Now $\nabla\cdot\overline{B} = \nabla\cdot\left[\oint \frac{\mu_0 I}{4\pi} \frac{d\overline{L}\times\overline{a_R}}{R^2}\right]$

$$= \frac{\mu_0 I}{4\pi}\oint \frac{\nabla\cdot(d\overline{L}\times\overline{a_R})}{R^3}$$

and we know that

$$\nabla\cdot(A\times B) = \overline{B}\cdot\nabla\times\overline{A} - \overline{A}\cdot\nabla\times\overline{B}$$

$$\therefore \nabla\cdot\overline{B} = \frac{\mu_0 I}{4\pi}\oint\left[\frac{\overline{a_R}}{R^2}\cdot(\nabla\times d\overline{L}) - d\overline{L}\cdot\left(\nabla\times\frac{\overline{a_R}}{R^2}\right)\right]$$

Since R is only in radial direction $\qquad a_R = R\overline{a}_\rho$

$\frac{\overline{a_R}}{R^2} = \frac{\overline{a_\rho}}{R^2} = -\nabla\left(\frac{1}{R}\right)\overline{a}_\rho \qquad \therefore \frac{\overline{a_R}}{R^2} = \frac{R\overline{a_\rho}}{R^2} = \frac{1}{R}\overline{a}_\rho$

$$\therefore \nabla\cdot\overline{B} = \frac{\mu_0 I}{4\pi}\oint\left[-\nabla\frac{1}{R}\overline{a}_\rho(\nabla\times d\overline{L}) + d\overline{L}\cdot\left(\nabla\times\nabla\left(\frac{-1}{R}\right)\overline{a}_\rho\right)\right]$$

let $\frac{1}{R}\overline{a}_\rho = \overline{C}$ and $d\overline{L} = \overline{D}$

Then $\nabla\cdot\overline{B} = \frac{\mu_0 I}{4\pi}\oint\left[-\nabla\overline{C}\cdot(\nabla\times\overline{D}) + D\cdot(\nabla\times\nabla C)\right]$

From the vector identity $\nabla\times\nabla C = 0$ and

$\nabla\cdot\nabla\times D = 0$

$\therefore \boxed{\nabla\cdot\overline{B} = 0}$ ⟶ Maxwell's Equation for Magneto-statics.

Since $\nabla\cdot\overline{B} = 0$, the magnetic flux lines are continuous.

Magnetic Scalar and Vector Potentials

We recall that some electrostatic field problems were simplified by relating the electric potential V to the electric field intensity $\bar{E}$ ($\bar{E} = -\nabla V$). Similarly, we can define a potential associated with magnetostatic field $\bar{B}$. In fact, the magnetic potential could be scalar V_m or vector $\bar{A}$. To define scalar and vector magnetic potentials, let us use two vector identities which are listed as the properties of curl:

$$\left.\begin{array}{ll} \nabla \times (\nabla V) = 0 & V = \text{Scalar} \\ \nabla \cdot (\nabla \times \bar{A}) = 0 & \bar{A} = \text{Vector} \end{array}\right\} \longrightarrow ①$$

- Every scalar V and vector $\bar{A}$ must satisfy these identities.

Just as $\bar{E} = -\nabla V$, we define the magnetic scalar potential V_m (in Amps) as related to H according to

$$\boxed{\bar{H} = -\nabla V_m} \quad \text{if } \bar{J} = 0 \longrightarrow ②$$

$\nabla \times E = 0$
$\oint \bar{E} \cdot d\bar{l} = 0$
$V_{ab} = -\int_a^b \bar{E} \cdot d\bar{l}$
Similarly $\nabla \times \bar{H} = 0$ if $\bar{J} = $
but $\oint \bar{H} \cdot d\bar{l} = I$
if no current is enclosed then

$$\boxed{V_{m_{ab}} = -\int_b^a \bar{H} \cdot d\bar{l}}$$ for a specific path

and we know

$$\bar{J} = \nabla \times H \longrightarrow ③$$

Substituting ② into equation ③

$$\bar{J} = \nabla \times (-\nabla V_m) = 0$$

Since V_m must satisfy the condition in eq ①. Thus the magnetic scalar potential V_m is only defined in a region where $\bar{J} = 0$ as in equation ②. We should also note that V_m satisfies Laplace's equation just as V does for electrostatic fields; hence

Laplace's equation for scalar magnetic potential $\longrightarrow$ $$\boxed{\nabla^2 V_m = 0 \quad (\bar{J} = 0)} \longrightarrow ④$$

$\oint \bar{B} \cdot d\bar{S} = 0$ using divergence theorem
$\oint \bar{B} \cdot d\bar{S} = \int (\nabla \cdot \bar{B}) dv = 0$
$\nabla \cdot \bar{B} = 0 \Rightarrow \nabla \cdot (\mu_0 H) = 0 \quad \mu_0 \neq$
$\nabla \cdot H = 0$
$\nabla \cdot (-\nabla V_m) = 0$
$\boxed{\nabla^2 V_m = 0}$ for $\bar{J} = 0$

We know that for a magnetic field $\nabla \cdot \bar{B} = 0 \longrightarrow ⑤$

To satisfy eq ⑤ and equation ①, we can define the vector magnetic potential $\bar{A}$ (Wb/m) such that

$$\boxed{\bar{B} = \nabla \times \bar{A}}$$

$\because \nabla \cdot \nabla \times \bar{A} = 0$, $\nabla \cdot B = 0$ $\Rightarrow$ $\nabla \cdot \nabla \times \bar{A} = \nabla \cdot \bar{B}$
$\therefore \nabla \times \bar{A} = \bar{B}$

Thus the curl of vector magnetic potential is the flux density

$\psi = \int \bar{B} \cdot d\bar{S} = \int_S (\nabla \times \bar{A}) \cdot d\bar{S}$
$= \oint_L \bar{A} \cdot d\bar{l}$

$$\boxed{\psi = \oint_L \bar{A} \cdot d\bar{l}}$$

Now $\nabla \times \bar{H} = \bar{J}$

$\therefore \nabla \times \frac{\bar{B}}{\mu_0} = \bar{J} \quad \because \bar{B} = \mu_0 \bar{H}$

$\nabla \times \bar{B} = \mu_0 \bar{J} \quad \because \bar{B} = \nabla \times \bar{A}$

$\nabla \times \nabla \times \bar{A} = \mu_0 \bar{J}$

Using vector identity to express LHS, we can write

$$\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} = \mu_0 \bar{J}$$

$\boxed{\nabla^2 A = -\mu_0 J}$ — poisson's eq

$V = \int \frac{dQ}{4\pi\epsilon_0 r}$

$A = \int \frac{\mu_0 I \, d\bar{l}}{4\pi R}$ line current

$A = \int \frac{\mu_0 \bar{K} dS}{4\pi R}$ Surface curr

$$\boxed{\bar{J} = \frac{1}{\mu_0}\left[\nabla \times \nabla \times \bar{A}\right] = \frac{1}{\mu_0}\left[\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}\right]}$$

Thus if vector magnetic potential is known, then current density J can be obtai

Scalar Magnetic potential :

Def: the line integral of the magnetic field intensity along a given path, where the current density is zero.

$$V_m = -\int \bar{H}\cdot d\bar{l}$$

between two points $V_{m A,B} = -\int_{A}^{B} \bar{H}\cdot d\bar{l}$

The negative gradient of the scalar magnetic potential is equal to the magnetic field intensity at which the current density $\bar{J}=0$

that is $\bar{H} = -\nabla V_m$ if $\bar{J}=0$

V_m is the scalar magnetic potential, units Amperes.

Taking curl on both sides

$$\nabla\times\bar{H} = -\nabla\times\nabla V_m = 0$$ (∵ curl of gradient of any scalar is zero)

From amperes law, $\nabla\times\bar{H} = \bar{J}$

$\therefore \quad \bar{J} = 0$

∴ the scalar magnetic potential V_m exists in the region where current density is zero.

The laplaces equation for scalar magnetic potential.

we know that $\nabla\cdot\bar{B} = 0$ or $\nabla\cdot\mu\bar{H} = 0$ since $\mu \neq 0$

$\nabla\cdot\bar{H} = 0$ ∴

But $\bar{H} = -\nabla V_m$

$\therefore \nabla(-\nabla V_m) = 0$

$\nabla^2 V_m = 0$ for $\bar{J} = 0$.

Note Scalar magnetic potential is used find the magnetic field intensity and flux in a region where there is no current exists. This happens in case of fields due to permanent magnets.

Vector Magnetic potential $\bar{A}$

In a given magnetic field, the curl of the vector magnetic potential $\bar{A}$ is equal to the magnetic flux density

$$\bar{B} = \nabla \times \bar{A}$$

where $\bar{A}$ is the vector magnetic potential Wb/m

poisson's Eq for $\bar{A}$

from Ampere's law

$$\nabla \times \bar{H} = \bar{J}$$

$$\nabla \times \frac{\bar{B}}{\mu} = J \text{ (or) } \nabla \times \bar{B} = \mu \bar{J}$$

But $\bar{B} = \nabla \times \bar{A}$

$$\therefore \nabla \times \nabla \times \bar{A} = \mu \bar{J}$$

$$\nabla \times \nabla \times \bar{A} = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} = \mu J$$

~~Since $\nabla \cdot \bar{B} = 0$~~ ~~$\nabla \cdot \nabla \times \bar{A} = 0$~~

From the vector identities $\nabla(\nabla \cdot \bar{A}) = 0$

$$\therefore -\nabla^2 \bar{A} = \mu \bar{J}$$ ~~or J = 0~~

$$\boxed{\nabla^2 \bar{A} = -\mu \bar{J}}$$ poisson's Eq

If $\bar{J} = 0$

then $\boxed{\nabla^2 \bar{A} = 0}$ (Laplace's Eq)

The vector magnetic potential is used to ~~find~~ obtain radiation characteristics of antennas.

$\overline{A}$ due to Differential Current element

Consider a differential element $d\overline{L}$ carrying current I. Then according to Biot-Savart's law the vector magnetic potential $\overline{A}$ at a distance R from the differential current element is given by

$$\overline{A} = \oint_L \frac{\mu_0 I \, d\overline{L}}{4\pi R} \text{ wb/m} \quad \text{— differential current element}$$

line current density

$$\overline{A} = \oint_S \frac{\mu_0 \overline{K} ds}{4\pi R} \text{ wb/m} \quad \text{— Surface current density}$$

$$\overline{A} = \oint_{vol} \frac{\mu_0 \overline{J} dv}{4\pi R} \text{ wb/m} \quad \text{— volume current density.}$$

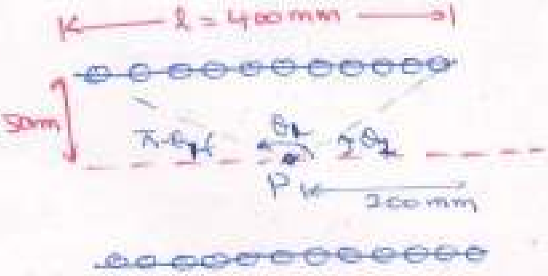
Q: A uniform solenoid 100 mm in diameter and 400 mm long has 100 turns of wire and a current of $I = 3A$. Find magnetic flux density $\overline{B}$ on the axis of the solenoid (a) at the centre (b) at one end, and (c) half way from the centre to one end.

Given data: $d = 100mm = 100 \times 10^{-3} m = 0.1m$

$l = 400mm = 400 \times 10^{-3} m = 0.4m$

$N = 100$ turns

$I = 3A$



(a) $\overline{B}$ at the centre of the axis.

Magnetic flux density at any point $\overline{P}$ on the axis

$$\overline{B} = \frac{\mu_0 N I}{2l}\left[\cos\theta_2 - \cos\theta_1\right]$$

From the figure $\tan\theta_2 = \frac{50 \times 10^{-3}}{200 \times 10^{-3}} = 0.25$

$$\theta_2 = \tan^{-1}(0.25) = 14.036^\circ$$

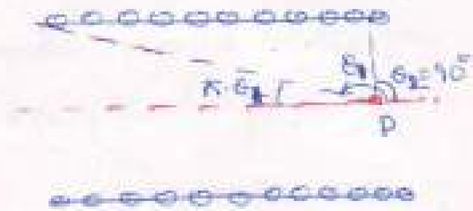
& $\tan(\pi - \theta_1) = \frac{50 \times 10^{-3}}{200 \times 10^{-3}} = 0.25$

$$\pi - \theta_1 = \tan^{-1}(0.25)$$
$$= 14.036^\circ$$
$$\therefore \theta_1 = 180 - 14.036^\circ = 165.964^\circ$$

$$\therefore |\overline{B}| = \frac{4\pi \times 10^{-7}}{2} \times \frac{100}{400 \times 10^{-3}} \times 3\left[\cos(14.036 - \cos 165.964)\right]$$

$$= \frac{9.425 \times 10^{-7} \times 10^{3}}{2}\left[0.97 - (-0.97)\right]$$

$|\overline{B}| = 0.914$ mWb/m²



B. $\overline{B}$ at one end.

$\theta_2 = 90^\circ$ and

$\tan(\pi - \theta_1) = \frac{50 \times 10^{-3}}{400 \times 10^{-3}} = 1/8$

$\pi - \theta_1 = 7.125^\circ$

$\theta_1 = 172.87^\circ$

$$\therefore B = \frac{\mu_0 n I}{2}\left[\cos\theta_2 - \cos\theta_1\right] = \frac{9.425 \times 10^{-7} \times 10^{3}}{2}\left[\cos 90^\circ - \cos 172.87^\circ\right]$$

$$= \frac{9.425 \times 10^{-7} \times 10^{3}}{2}\left[0 - (-0.9922)\right]$$

$= 0.467$ mWb/m²

c. $\overline{B}$ half way from the centre to one end.

$\tan\theta_2 = \frac{50}{100} = 0.5$

$\theta_2 = 26.56^\circ$

$\tan(\pi-\theta_1) = \frac{50}{200} = \frac{1}{6} = 0.166$

$\pi-\theta_1 = 9.458$

$\theta_1 = 170.54^\circ$

$\therefore B = 9.425\times10^{-7}\times10^{3}[\cos 26.56^\circ - \cos 170.54^\circ]$

$= 0.886\ \text{mwb}/\text{m}^2$

eg: A steady current I amps flow in a conductor bent in the form of hexagon. Find the intensity at the centre of the loop. The distance between the centre and each side is 'a' meters.

Let H_1 be the magnetic field intensity due to one side

$\therefore H_1 = \frac{I}{4\pi\rho}[\cos\alpha_2 - \cos\alpha_1]$

$= \frac{I}{4\pi a}[\cos 60^\circ - \cos(180-60^\circ)]$

$= \frac{I}{4\pi a}[0.5+0.5] = \frac{I}{4\pi a}$

H due to Hexagone is six times the field intensity due to each conductor

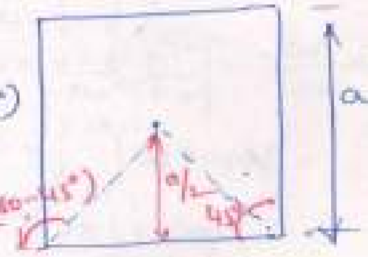
$\therefore H = H_1+H_2+H_3+H_4+H_5+H_6$

$= 6H$

$= 6\times\frac{I}{4\pi a}$

$\therefore H = \frac{1.5I}{\pi a}$

eg: A steady current of I amperes flow in a conductor bent in the form of square loop of side 'a'. find H at the centre of the loop.



$H_1 = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)$

$= \frac{I}{4\pi a/2}[\cos 45 - \cos 135^\circ) = \frac{I}{2\pi a}(\cos 45^\circ - (-\cos 45^\circ))$

$= \frac{I}{2\pi a}2\cos 45^\circ$

$= \frac{I}{\sqrt{2}\pi a}$

$\therefore H = 4H_1 = 4\times\frac{I}{\sqrt{2}\pi a}$

$H = \frac{2\sqrt{2}I}{\pi a}$

Eg: Given the magnetic vector potential $\overline{A} = -\rho^2/4\ \bar{a}_z$ wb/m, calculate the total flux crossing the surface $\phi = \pi/2$, $1 \le \rho \le 2$ m, $0 \le z \le 5$ m.

Method 1.

$$\overline{B} = \nabla \times \overline{A} = \frac{-\partial A_z}{\partial \rho}\bar{a}_\phi = \frac{\rho}{2}\bar{a}_\phi$$

$$d\bar{s} = d\rho\, dz\, \bar{a}_\phi$$

Hence,

$$\psi = \int \overline{B}\cdot d\bar{s} = \frac{1}{2}\int_{z=0}^{5}\int_{\rho=1}^{2} \rho\, d\rho\, dz$$

$$= \frac{1}{4}\rho^2\Big|_2^1 (5) = \frac{15}{4}$$

$\therefore \psi = 3.75$ wb.

Method 2

$$\psi = \oint_L \overline{A}\cdot d\bar{L} = \psi_1 + \psi_2 + \psi_3 + \psi_4$$

Since $\overline{A}$ has only a z-component

$$\psi_1 = \psi_3 = 0$$

$$\therefore \psi = \psi_2 + \psi_4 =$$

$$= -\frac{1}{4}\left[(1)^2\int_0^5 dz + (2)^2\int_5^0 dz\right]$$

$$= -\frac{1}{4}(1-4)(5) = \frac{15}{4} = 3.75\text{ wb.}$$

Eg: Find V_m for the region within the toroid with a rectangular crosssection enclosed by planes $z = 0$ & $z = 2$ cm, cylinders $\rho = 5$ cm & $\rho = 7$ cm. $\overline{H} = \frac{30}{\rho}\bar{a}_\phi$ A/m and $V_m = 0$ at $\rho = 6$ cm & $\phi = 0.6$ rad, $z = 2$ cm. Keep ϕ within the range of $0 \le \phi \le 2\pi$.

Sol:- The toroid has $H = \frac{30}{\rho}\bar{a}_\phi$ A/m

The dimensions are

$$0.05 \le \rho < 0.07 \ \& \ 0 < z < 0.03$$

$$\overline{H} = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}\bar{a}_\phi \Rightarrow \frac{30}{\rho} = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$

$$V_m = \cancel{-30\phi}\ \cancel{-30\phi} - 30\phi + C_1$$

The boundary conditions are

$V_m = 0$ at $\rho = 0.06$ m & $\phi = 0.6$

& $z = 0.02$ m

$$\therefore \quad 0 = -30\times 0.6 + C_1$$

$$\Rightarrow C_1 = 18$$

$\therefore$ Magnetic Scalar potential

$$V_m = -30\phi + 18$$

Eg: If $\overline{A} = 10\rho^{3/2}\bar{a}_z$ wb/m in free space. Find (a) $\overline{H}$, (b) $\overline{J}$; and (c) show that $\oint \overline{H}\cdot d\overline{L} = I$ for a circular path with $\rho = 1$

Sol:

$$\overline{B} = \nabla\times\overline{A} = \begin{vmatrix} \frac{\bar{a}_\rho}{\rho} & \bar{a}_\phi & \frac{\bar{a}_z}{\rho} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & 0 & \boxed{A_z} \end{vmatrix} \leftarrow 10\rho^{3/2} = -\bar{a}_\phi\frac{\partial}{\partial\rho}(10\rho^{3/2})$$

$$\therefore \overline{B} = -\frac{\partial}{\partial\rho}(10\rho^{3/2})\bar{a}_\phi = -15\rho^{1/2}\bar{a}_\phi \text{ T}$$

$$\therefore (a)\ H = \frac{B}{\mu_0} = -\frac{15}{\mu_0}\sqrt{\rho}\,\bar{a}_\phi \text{ A/m}$$

$$(b)\ \overline{J} = \nabla\times\overline{H} = \begin{vmatrix} \frac{\bar{a}_\rho}{\rho} & \bar{a}_\phi & \frac{\bar{a}_z}{\rho} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & \rho H_\phi & 0 \end{vmatrix} \longrightarrow -\frac{15}{\mu_0}\rho^{3/2}$$

$$= \frac{a_z}{\rho}\frac{\partial}{\partial\rho}\left(-\frac{15}{\mu_0}\rho^{3/2}\right)$$

$$J = -\frac{15}{\rho\mu_0}\bar{a}_z \times 1.5\rho^{1/2} = \frac{-22.5\sqrt{\rho}}{\mu_0}\bar{a}_z \text{ A/m}^2$$

(c) Current enclosed in $\rho = 1$

$$I = \int_0^{2\pi}\int_0^1 |\overline{J}|\rho\, d\rho\, d\phi$$

$$= \frac{22.5}{\mu_0}\int_0^{2\pi}\int_0^1 \sqrt{\rho}\, d\rho\, d\phi$$

$$= -2\pi\frac{22.5}{\mu_0}\frac{\rho^{3/2}}{1.5}\Big|_0^1 = -\frac{30\pi}{\mu_0}$$

&

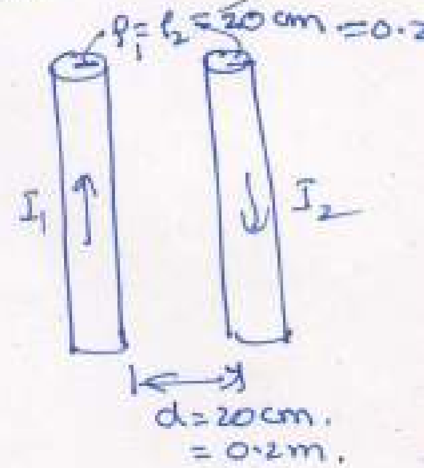
$$\oint H dL = \int_0^{2\pi}\int_0^1 H_\phi\, \rho\, d\rho\, d\phi$$

$$= -\frac{15}{\mu_0}2\pi\int_0^1 \rho\sqrt{\rho}\, d\rho =$$

$$= -\frac{15}{\mu_0}2\pi\frac{\rho^{2.5}}{2.5}\Big|_0^1 = -\frac{30\pi}{\mu_0}\text{ A}$$

$$\therefore \oint \overline{H}\cdot d\overline{L} = I.$$

Currents $I_1 = I_2 = 10A$ flowing in opposite direction through two long parallel wires separated by a distance of 20cm. Find the magnitude and the direction of magnetic flux density at a point 20 cm away from each other. (May/June 2009)



Sol: $I_1 =$ ~~100~~ $10A$ — Conductor 1.

$I_2 = -10A$ — Conductor 2.

Magnetic flux density in conductor 1

$$B_1 = \frac{\mu I_1}{2\pi\rho_1} = \frac{\mu}{2\pi} \times \frac{100}{0.2}$$

and conductor 2

$$B_2 = \frac{\mu I_2}{2\pi\rho_2} = \frac{-\mu}{2\pi} \times \frac{100}{0.2}$$

∴ Total magnetic flux density

$$B = B_1 + B_2$$

$$= \frac{\mu}{2\pi} \times \frac{100}{0.2} - \frac{\mu}{2\pi} \times \frac{100}{0.2}$$

$$= 0.$$

UNIT V : AMPERE'S LAW or AMPERE'S CIRCUIT LAW

Ampere's law similar to Gauss's law is used to obtain Magnetic field Intensity $\bar{H}$ when current distribution is symmetrical.

Definition: Ampere's Circuit law states that the line integral of $\bar{H}$ around a closed path is the same as the net current I_{enc} enclosed by the path

$$\boxed{\oint \bar{H}\cdot d\bar{L} = I_{enc}} \longrightarrow ①$$

Integral form

Note: Equation above always holds regardless of whether the current distribution is symmetrical, but we can use the equation to determine $\bar{H}$ only when a symmetrical current distribution exists.

This law is very useful to determine $\bar{H}$ when the current distributi is symmetrical.

Ampere's law is a special case of Biot-Savart's Law, similar to the Gauss's Law is a special case of Coulomb's law. The former may be derived from the latter.

The application of Gauss's law involves finding the total charge enclosed by a closed surface: while the application of Ampere's law involves finding the total current enclosed by a closed pat

By applying Stoke's Theorem to the above equation, we obtain

$$I_{enc} = \oint_L \bar{H}\cdot d\bar{L} = \int_S (\nabla\times\bar{H})\cdot d\bar{S} \longrightarrow ②$$

But $I_{enc} = \int_S \bar{J}\cdot d\bar{S} \longrightarrow ③$ Total Current flowing through a Surface S. (2nd unit Convecti & Conduction Currents) J – Current density.

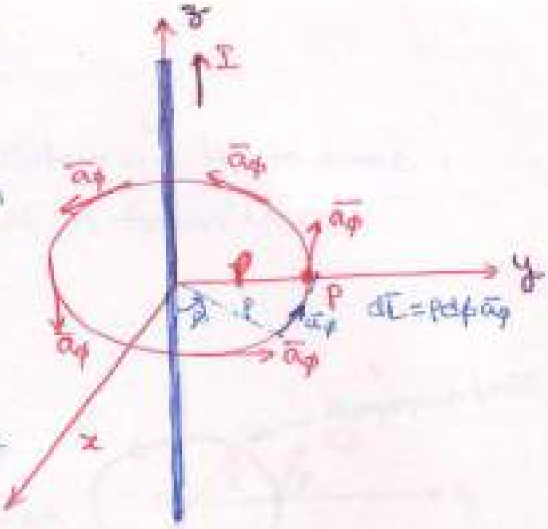
Comparing equations ② & ③

$$\boxed{\nabla\times\bar{H} = \bar{J}}$$

Maxwell's 2nd Equation
Ampere's law in differential form
or point form

We should observe that $\nabla\times\bar{H} = \bar{J} \neq 0$; that is a magnetostati field is not Conservative.

Ampere's Circuit Law (Proof)



Consider a long straight conductor carrying direct current I placed along z-axis as shown in figure. Consider a closed circular path of radius 'r' which encloses the straight conductor carrying direct current I.

The point P is at a perpendicular distance ρ from the conductor. Consider $d\bar{L}$ at point P which is in $\bar{a}_\phi$ direction tangential to the circular path at P.

$$d\bar{L} = \rho d\phi \bar{a}_\phi$$

While $\bar{H}$ obtained at point P, from Biot-Savart's law due to infinitely long conductor is,

$$\bar{H} = \frac{I}{2\pi\rho} \bar{a}_\phi$$

$$\therefore \bar{H} \cdot d\bar{L} = \frac{I}{2\pi\rho} \bar{a}_\phi \cdot \rho d\phi \bar{a}_\phi$$

$$= \frac{I}{2\pi\rho} \rho d\phi = \frac{I}{2\pi} d\phi \qquad (\because \bar{a}_\phi \cdot \bar{a}_\phi = 1)$$

Integrating the above equation over the entire closed path

$$\oint \bar{H} \cdot d\bar{L} = \int_{\phi=0}^{2\pi} \frac{I}{2\pi} d\phi = \frac{I}{2\pi} [\phi]_0^{2\pi} = \frac{I \, 2\pi}{2\pi}$$

$$= I \quad \text{Current carried by the conductor.}$$

This proves that the integral $\bar{H} \cdot d\bar{L}$ along the closed path gives the direct current enclosed by that closed path.

Note:- The path enclosing the direct current I need not be a circular and it may be any irregular shape. The law does not depend on the shape of the path but the path must be enclosed the direct current once. This path selected is called Amperian path similar to the Gaussian surface used while applying Gauss's law.

* Ampere's law is analogous to Gauss' law in electrostatics.

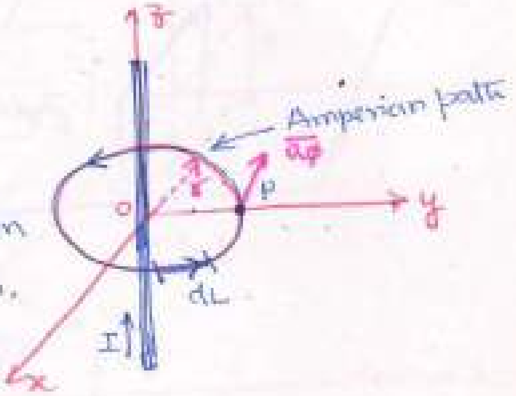
* For a conductor of cross-section $S \, m^2$ with current density $J \, A/m^2$, the Ampere's law can be expressed as

$$I_{enc} = \oint \bar{H} \cdot d\bar{L} = \int_S \bar{J} \cdot d\bar{S}$$

Applications of Ampere's Law.

We now apply Ampere's Circuit law to determine $\overline{H}$ for some symmetrical current distributions as we did for Gauss' law.

A. Infinite Line Current



Consider an infinite long straight conductor placed along z-axis, carrying a direct current I as shown in figure. Consider an Amperian closed path, enclosing the conductor as shown. Consider a point P on the closed path at which $\overline{H}$ is to be obtained. The radius of the path is ρ and hence P is at perpendicular distance ρ from the conductor.

The magnitude of $\overline{H}$ depends on ρ and the direction is always tangentia
to the closed path i.e. $\overline{a_\phi}$. So $\overline{H}$ has only component in $\overline{a_\phi}$ directio
say ~~$\overline{H}_\phi$~~ H_ϕ $\overline{H} = H_\phi \overline{a_\phi}$

Consider elementary length $d\overline{L}$ at point P. In cylindrical coordinat

$$d\overline{L} = \rho\, d\phi\, \overline{a_\phi}$$

∴ ~~$\overline{H} = H_\phi \overline{a_\phi}$~~

$$\text{and } \overline{H} \cdot d\overline{L} = H_\phi \overline{a_\phi} \cdot \rho\, d\phi\, \overline{a_\phi}$$

$$= H_\phi\, \rho\, d\phi$$

According to Ampere's Circuit law,

$$\oint \overline{H} \cdot d\overline{L} = I$$

$$\therefore \int_{\phi=0}^{2\pi} H_\phi\, \rho\, d\phi = I$$

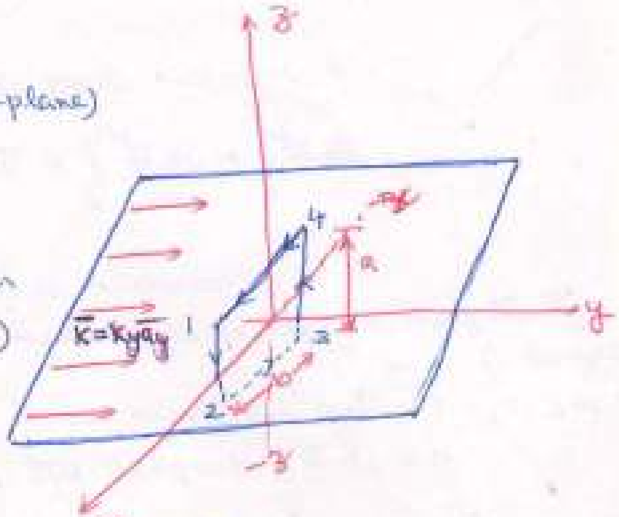
$$\rho H_\phi \int_{\phi=0}^{2\pi} d\phi = I$$

$$H_\phi = \frac{I}{2\pi\rho}$$

Hence $\overline{H}$ at point P is given by

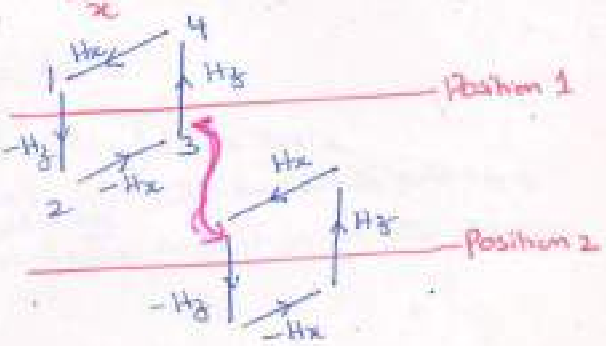
$$\boxed{\overline{H} = H_\phi \overline{a_\phi} = \frac{I}{2\pi\rho} \overline{a_\phi} \quad A/m}$$

B. INFINITE SHEET OF CURRENT



Consider an infinite current sheet in (xy-plane) $z=0$ plane. If the sheet has a uniform current density $\overline{K}=K_y\overline{a}_y$ A/m, applying Ampere's Law to the rectangular closed path 1-2-3-4-1 (Amperian path) gives

$$\oint \overline{H}\cdot d\overline{L} = I_{enc} = K_y b \longrightarrow ①$$



From the figure, it is clear that in between two very closely spaced conductors, the components of $\overline{H}$ in z-direction are oppositely directed ($-H_z$ for position 2 & H_z for position 1).

All such components cancel each other and hence $\overline{H}$ can not have any component in z-direction.

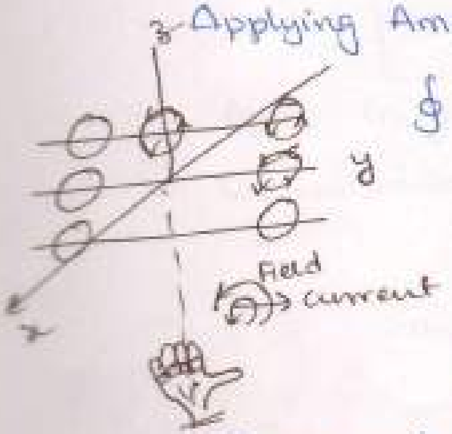
As current is flowing in y direction, $\overline{H}$ can not have component in y direction.

So $\overline{H}$ has only component in x direction.

$$\therefore\ \overline{H} = H_x\overline{a_x} \quad \text{for } z>0$$
$$= -H_x\overline{a_x} \quad \text{for } z<0 \Big\} \longrightarrow ②$$

$$\int_1^2 \overline{H}\cdot d\overline{L} = \int_1^2 H_z\overline{a}_x \cdot dz\overline{a}_z = 0$$

$$\int_2^3 \overline{H}\cdot d\overline{L} = \int_2^3 -H_x\overline{a}_x\cdot dx\overline{a}_x = -H_x\int_2^3 dx = -H_x(-b) = H_x b$$



Applying Ampere's Law along the closed path

$$\oint \overline{H}\cdot d\overline{L} = \int_1^2 \overline{H}\cdot d\overline{L} + \int_2^3 \overline{H}\cdot d\overline{L} + \int_3^4 \overline{H}\cdot d\overline{L} + \int_4^1 \overline{H}\cdot d\overline{L}$$

$$= -H_z(-a) + (-H_x)(-b) + H_z(a) + H_x(b)$$

$$= 0(-a) + (-H_x)(-b) + 0(a) + H_x(b)$$

$$\oint \overline{H}\cdot d\overline{L} = 2H_x b \longrightarrow ③$$

From equations ① & ③ $H_x = \frac{1}{2}K_y$ Substituting into eq ②

$$\therefore\ \boxed{\begin{aligned}\overline{H} &= \tfrac{1}{2}K_y\,\overline{a}_x \quad z>0 \\ &= -\tfrac{1}{2}K_y\,\overline{a}_x \quad z<0\end{aligned}}$$

In general $\boxed{\overline{H} = \frac{1}{2}\overline{K}\times\overline{a}_n}$

Alternative method.

Applying Ampere's Circuit law for the path 1-2-3-4-1

$$\oint \overline{H}\cdot d\overline{L} = \int_1^2 \overline{H}\cdot d\overline{L} + \int_2^3 \overline{H}\cdot d\overline{L} + \int_3^4 \overline{H}\cdot d\overline{L} + \int_4^1 \overline{H}\cdot d\overline{L}$$

for path 1-2 $\quad d\overline{L} = dz\,\overline{a}_z$

for path 3-4 $\quad d\overline{L} = dz\,\overline{a}_z$

$\Big\}$ $\overline{H}$ is in x direction

$\therefore \overline{H}\cdot d\overline{L} = H_x a_x \cdot dz\, a_z = H_x dz\, \overline{a}_x\cdot\overline{a}_z$
$= 0 \quad (\because a_x\cdot a_z = 0)$

for path 2-3 $\quad dL = dx\,\overline{a}_x$

for path 4-1 $\quad dL = dx\,\overline{a}_x$

Hence along the paths 1-2, & 3-4 the integral $\oint \overline{H}\cdot dL = 0$.

Consider path 2-3 along which $d\overline{L} = dx\,\overline{a}_x$

$$\therefore \int_2^3 \overline{H}\cdot d\overline{L} = \int_2^3 (-H_x a_x)\cdot(dx\,\overline{a}_x) = H_x\int_2^3 dx = bH_x$$

the path 2-3 is lying in $z<0$ region for which $\overline{H}$ is $-H_x\overline{a}_x$. And limits from 2 to 3, +ve x to −ve x hence effective sign of the integral is +ve.

Consider path 4-1 along which $d\overline{L} = dx\,\overline{a}_x$ and it is in the region $z>0$ hence $\overline{H} = H_x\overline{a}_x$

$$\therefore \int_4^1 \overline{H}\cdot d\overline{L} = \int_4^1 (H_x a_x)\cdot(dx\,\overline{a}_x) = H_x\int_4^1 dx = bH_x$$

$$\therefore \oint \overline{H}\, d\overline{L} = bH_x + bH_x = 2bH_x$$

Equating this to eq ①

$2bH_x = K_y b$

$H_x = \frac{1}{2}K_y$

Hence $\boxed{\begin{aligned}\overline{H} &= \tfrac{1}{2}K_y\,\overline{a}_x \quad \text{for } z>0 \\ &= -\tfrac{1}{2}K_y\,\overline{a}_x \quad \text{for } z<0\end{aligned}}$

Again in general, for an infinite sheet of current density $\overline{K}$ A/m we can write,

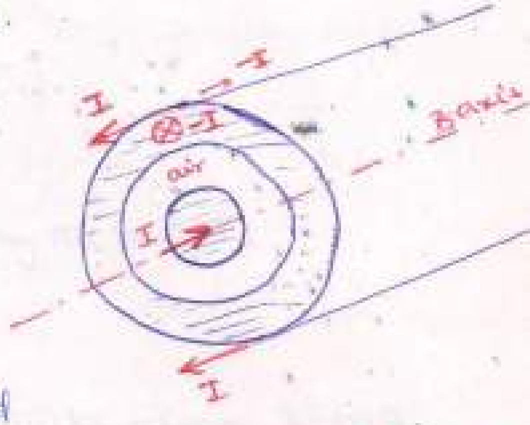
$$\boxed{\overline{H} = \frac{1}{2}\overline{K}\times\overline{a}_n}$$

$\overline{a}_n$ – unit vector normal from the current sheet to the point at which $\overline{H}$ is to be obtained.

C. Infinitely Long Coaxial Transmission Line.

Consider an infinitely long transmission line consisting of two concentric cylinders having their axes along z-axis.

Let the inner conductor has a radius a and carries a current I, while the outer conductor has inner radius b and thickness t and carries return current -I.



Now inorder to determine $\bar{H}$ everywhere, assume that current is uniformly distributed in both the conductors. Since the current distribution is symmetrical, we apply Ampere's law along the Amperian path for each of four possible regions; $0 \le \rho \le a^{(L_1)}$, $a \le \rho \le b^{(L_2)}$, $b \le \rho \le c$ (or b+t)$^{(L_3)}$ and $\rho > c$ (or) b+t(L_4)

Cross-section

Region 1 : For region 1 $0 \le \rho \le a$, we apply Ampere's law to path L_1, giving

$$\oint_{L_1} \bar{H} \cdot d\bar{l} = I_{enc} = \int \bar{J} \cdot d\bar{s}$$

Since the current is uniformly distributed over the cross-section,

Current per Area of cross section $= \bar{J} = \frac{I}{\pi a^2} \bar{a}_z \qquad d\bar{s} = \rho \, d\phi \, d\rho \, \bar{a}_z$

$$I_{enc} = \int \bar{J} \cdot d\bar{s} = \frac{I}{\pi a^2} \int_{\phi=0}^{2\pi} \int_{\rho=a}^{\rho} \rho \, d\phi \, d\rho$$

$$= \frac{I}{\pi a^2} \pi \rho^2 = \frac{I \rho^2}{a^2}$$

$$\therefore \oint_{L_1} \bar{H} \cdot d\bar{l} = H_\phi \int_{L_1} dL = H_\phi \, 2\pi\rho = \frac{I\rho^2}{a^2}$$

$$\text{or} \quad H_\phi = \frac{I\rho}{2\pi a^2}$$

$$\boxed{\therefore \bar{H} = \frac{I\rho}{2\pi a^2} \bar{a}_\phi \quad A/m}$$

$J = \pi a^2$

$I_{enc} = \pi \rho^2$

Alternatively

The area of cross section enclosed is $\pi \rho^2 \, m^2$ ~~πa^2 m~~.

The total current flowing is I through area πa^2. Hence the current enclosed by the closed path is

fractional area current = fractional current enclosed.

fractional area

$$I_{enc} = \frac{\pi \rho^2}{\pi a^2} I = \frac{\rho^2}{a^2} I.$$

The $\bar{H}$ is only in $\bar{a}_\phi$ direction and depends only on ρ.

$$\therefore \bar{H} = H_\phi \bar{a}_\phi$$

$$\text{and} \quad d\bar{l} = \rho \, d\phi \, \bar{a}_\phi$$

$$\therefore \bar{H} \cdot d\bar{l} = H_\phi \bar{a}_\phi \cdot \rho \, d\phi \, \bar{a}_\phi = H_\phi \rho \, d\phi$$

According to Ampere's law,

$$\oint \bar{H} \cdot d\bar{l} = I_{enc} = \frac{\rho^2}{a^2} I$$

$$\int_{\phi=0}^{\pi} H_\phi \rho \, d\phi = \frac{\rho^2}{a^2} I$$

$$H_\phi \rho [2\pi] = \frac{\rho^2}{a^2} I$$

$$\therefore H_\phi = \frac{\rho^2}{2\pi \rho a^2} I = \frac{\rho}{2\pi a^2} I$$

$$\boxed{\therefore \bar{H} = \frac{I\rho}{2\pi a^2} \bar{a}_\phi \quad A/m.}$$

Region 2 : $a \le \rho \le b$, we use path L_2 as Amperian path

$$\therefore \oint_{L_2} \bar{H}\cdot d\bar{l} = I_{enc} = I$$

$$H_\phi \, 2\pi\rho = I$$

$$\text{or } H_\phi = \frac{I}{2\pi\rho}$$

Since the whole current is enclosed by the path L_2

$$\boxed{\therefore \bar{H} = \frac{I}{2\pi\rho}\bar{a}_\phi \quad A/m}$$

This is the case for infinitely long conductor along z-axis.

Region 3 : $b \le \rho \le c$ (or $b+t$), we use path L_3, getting

$$\oint_{L_3} \bar{H}\cdot d\bar{l} = H_\phi \, 2\pi\rho = I_{enc}$$

where $I_{enc} = I + \int \bar{J}\cdot d\bar{S}$

J in this case is the current density (current per unit area) of the outer conductor and is along $-\bar{a}_z$

$$\therefore \bar{J} = \frac{-I}{\pi[(b+t)^2 - b^2]}\bar{a}_z$$

$$\therefore I_{enc} = I - \frac{I}{\pi[(b+t)^2 - b^2]}\int_{\phi=0}^{2\pi}\int_{\rho=b}^{\rho} \rho \, d\rho \, d\phi$$

$$= I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

Substituting into the above eq

$$H_\phi \cdot 2\pi\rho = I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

$$H_\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

$$\boxed{\therefore \bar{H} = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]\bar{a}_\phi \quad A/m}$$

Alternatively (Region 3)

Current enclosed by the closed path L_3 of outer conductor is

$$I'_{enc} = \frac{\pi(\rho^2 - b^2)}{\pi(c^2 - b^2)}(-I) = \frac{(\rho^2 - b^2)}{(c^2 - b^2)}(-I)$$

Note that the closed path also enclo the inner conductor hence the current I flowing through it.

$\therefore I''_{enc} = I$ = current in the inner conductor enclosed.

$\therefore$ Total current enclosed by the path

$$I_{enc} = I' + I'' = -\frac{(\rho^2 - b^2)}{(c^2 - b^2)} I + I$$

$$= I\left[\frac{c^2 - \rho^2}{c^2 - b^2}\right]$$

$\therefore$ According to Ampere's law

$$\oint \bar{H}\cdot d\bar{l} = I_{enc}$$

$$\oint \bar{H}\cdot d\bar{l} = \oint H_\phi \rho \, d\phi = I_{enc}$$

$$= I\left[\frac{c^2 - \rho^2}{c^2 - b^2}\right]$$

$$\therefore H_\phi \rho \, 2\pi = I\left[\frac{c^2 - \rho^2}{c^2 - b^2}\right]$$

$$\therefore H_\phi = \frac{I}{\rho 2\pi}\left(\frac{c^2 - \rho^2}{c^2 - b^2}\right)$$

$$\boxed{\therefore \bar{H} = \frac{I}{2\pi\rho}\left[\frac{c^2 - \rho^2}{c^2 - b^2}\right]\bar{a}_\phi \quad A/m}$$

Region 4 : $\rho > b+t$ or $\rho > c$

$$I_{enc} = +I - I = 0A$$

$$\therefore \oint \bar{H}\cdot d\bar{l} = 0$$

$$\boxed{\therefore \bar{H} = 0 \quad A/m}$$

$$\text{Hence } \overline{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2}\,\overline{a}_\phi & 0\le\rho\le a \\[2ex] \dfrac{I}{2\pi\rho}\,\overline{a}_\phi & \cancel{\rho}\; a\le\rho\le b \\[2ex] \dfrac{I}{2\pi\rho}\left[\dfrac{c^2-\rho}{c^2-b^2}\right]\overset{\cancel{\rho^2-b^2}}{\overline{a}_\phi} & b\le\rho\le c \\[2ex] 0 & \rho > c \end{cases}$$

MFI due to a Solenoid using Ampere's Circuit law.

$$\oint \bar{H}\cdot d\bar{l} = I_{encl}$$

For a uniform field, the MFI at points within the solenoid is

$$Hl = NI$$

l – length of solenoid.
N – no of turns

or $H = \frac{NI}{L}$ AT/m. along the axis.

If the solenoid is lying along the z-axis, then

$$\bar{H} = \frac{NI}{l}\bar{a}_z \quad AT/m$$

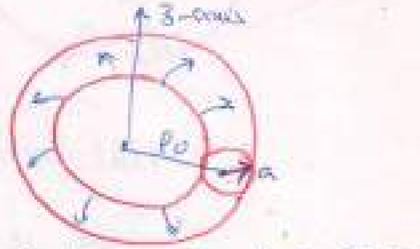
and $B = \frac{\mu NI}{l}\bar{a}_z$ Wb/m²

Toroid   MFI due to a Toroid

For a solenoid of a finite length l consists of N closely wound turns of a filament that carries a current I, then the field at points well within the solenoid is given closely by

$$\overline{H} = \frac{NI}{l}\,\overline{a_z} \quad \text{(well within the solenoid)}$$

For the toroid, the magnetic field for the ideal case



$K = K_a a_z,\ \rho = \rho_0 - a,\ z = 0$

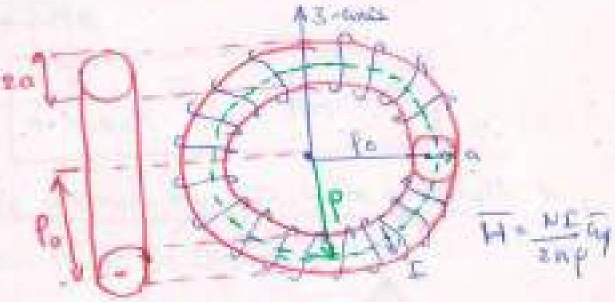
$$\overline{H} = K_a \frac{\rho_0 - a}{\rho}\,\overline{a_\phi} \quad \text{(inside toroid)}$$

$$\overline{H} = 0 \qquad \text{(outside toroid)}$$

For the N-turn toroid,

$$\overline{H} = \frac{NI}{2\pi\rho}\,\overline{a_\phi} \quad \text{(inside toroid)}$$

$$= 0 \qquad \text{(outside toroid)}$$



H on the axis of Toroid

A toroid whose dimensions are shown in figure has N turns and carries current I. To Determine H inside and outside the toroid, consider a path (Amperian path) which is a circle of radius ρ. Since N wires cut through the path each carrying current I, the net current enclosed by the Amperian path is NI, Hence

$$\oint \overline{H}\cdot d\overline{l} = I_{enc} \Rightarrow H \cdot 2\pi\rho = NI$$

$$\text{or } \boxed{H = \frac{NI}{2\pi\rho}} \quad \text{for } \rho_0 - a < \rho < \rho_0 + a$$

where ρ_0 is the mean radius of the toroid as shown in figure.

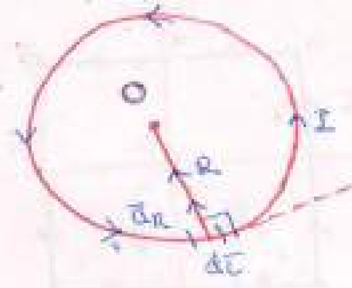
An approximate value of H is

$$\boxed{H_{approx} = \frac{NI}{2\pi\rho_0} = \frac{NI}{l}}$$

This is the same formula obtained for solenoid along its axis. $(\rho \gg a)$

Thus a straight solenoid may be regarded as a special toroidal coil for which $\rho_0 \to \infty$. Outside the toroid, the current enclosed by an Amperian path is $NI - NI = 0$ and hence $H = 0$.

$\overline{H}$ at the centre of the circle formed by the wire of length L



Wire of length L is formed into circle

$\therefore\ 2\pi R = L$ i.e. $R = \frac{L}{2\pi} = 0.1591L$

$d\overline{L}$ element is tangential to the circle hence is perpendicular to the radius $\overline{R}$

According to Biot-Savart's Law

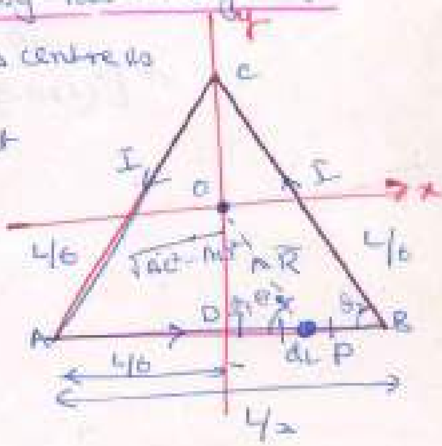
$$d\overline{H} = \frac{I d\overline{L} \times \overline{a}_R}{4\pi R^2} = \frac{I|d\overline{L}||\overline{a}_R|\sin\theta\, \overline{a}_N}{} \qquad (\because \theta = 90^\circ, |\overline{a}_R| = 1$$

$$d\overline{H} = \frac{I d\overline{L}\, \overline{a}_N}{4\pi R^2} \Rightarrow \overline{H} = \int d\overline{H} = \int \frac{I dL \overline{a}_N}{4\pi R^2} = \frac{I}{4\pi R^2}\int dL\ \overline{a}_N$$

Now $\int dL$ = Circumference of the circle $= 2\pi R$

$$\therefore \overline{H} = \frac{I \times 2\pi R \times \overline{a}_N}{2\pi R^2} = \frac{I}{2R}\overline{a}_N = \boxed{\frac{I}{0.3182L}\overline{a}_N\ A/m = \overline{H}}$$

$\overline{H}$ at the centre of an equilateral triangle formed by the wire of length L



Let the triangle placed in x-y plane such that its centre is at the origin.

Consider differential length dL at point P, which is at a distance x from D. The lengths $AC = \frac{L}{3}$, $AD = \frac{L}{6}$,

$CD = \sqrt{AC^2 - AD^2} = 0.2886L$

From the equilateral triangle $OD = \frac{1}{3} \times 0.2886L$
$= 0.0962L$

$\overline{R} = -x\overline{a}_x + 0.0962L\overline{a}_y$; $|\overline{R}| = \sqrt{x^2 + (0.0962L)^2}$

$$\therefore \overline{a}_R = \frac{\overline{R}}{|\overline{R}|} = \frac{-x\overline{a}_x + 0.0962L\overline{a}_y}{\sqrt{x^2 + (0.0962L)^2}}$$

$\cos\theta = \frac{x}{R}$

$\sin\theta = \frac{y}{R}$

~~$\tan\theta$~~ $= \frac{y}{}$

and $d\overline{L} = dx\,\overline{a}_x$

$$d\overline{L} \times \overline{a}_R = \begin{vmatrix} \overline{a}_x & \overline{a}_y & \overline{a}_z \\ dx & 0 & 0 \\ -x & 0.0962L & 0 \end{vmatrix} = \frac{0.0962L\,dx\,\overline{a}_z}{\sqrt{x^2 + (0.0962L)^2}}$$

$$d\overline{H} = \frac{I d\overline{L} \times \overline{a}_R}{4\pi R^2} = \frac{I \times 0.0962L\,dx\,\overline{a}_z}{4\pi[x^2 + (0.0962)^2]^{3/2}}$$

$$\therefore \overline{H} = \int_{x=-L/6}^{L/6} d\overline{H} = 2\int_{x=0}^{L/6} d\overline{H} = \frac{2 \times I \times 0.0962}{4\pi}\int_{x=0}^{x=L/6} \frac{dx}{\{x^2 + (0.0962L)^2\}^{3/2}}$$

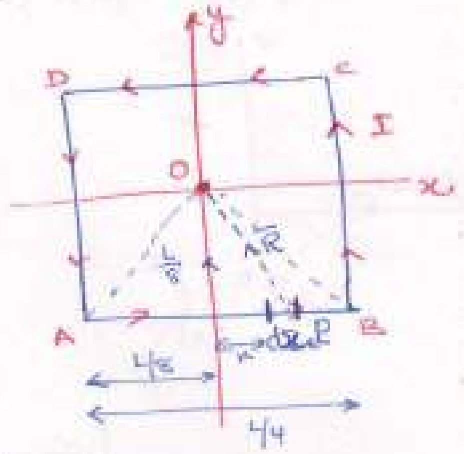
put $x = 0.0962\tan\theta$, $dx = 0.0962L\sec^2\theta\,d\theta$

for $x = 0, \theta_1 = 90$; and $x = L/6$ $\theta_2 = 60^\circ$

$$\therefore \overline{H} = \frac{I \times 0.0962L}{2\pi}\int_0^{60} \frac{0.0962L\sec^2\theta\,d\theta\,\overline{a}_z}{(0.0962L)^3\sec^3 d\theta} = \frac{1}{2\pi \times 0.0962L}[\sin]_0^{60} = \frac{1.4321}{L}\overline{a}_z$$

All the sides are producing $\overline{H}$ in the same direction. Hence total $\overline{H}$ at the origin i.e. centre is $\boxed{\overline{H}_{total} = 3\overline{H} = \frac{4.296I}{L}\overline{a}_z\ A/m}$

$\overline{H}$ at the centre of a Square formed by the wire of length L



Consider the square in x-y plane, such that centre is at the origin. Consider the differential length dx at P, at a distance of x. Thus

$$d\overline{L} = dx\,\overline{a_x}$$

$$\overline{R} = -x\overline{a_x} + \frac{L}{8}\overline{a_y} = -x\overline{a_x} + 0.125\,L\overline{a_y}$$

$$|\overline{R}| = \sqrt{x^2+(0.125)L]^2}$$

$$\therefore \overline{a_R} = \frac{-x\overline{a_x} + 0.125\,L\overline{a_y}}{\sqrt{x^2+(0.125L)^2}}$$

$$\therefore d\overline{L}\times\overline{a_R} = \begin{vmatrix} \overline{a_x} & \overline{a_y} & \overline{a_z} \\ dx & 0 & 0 \\ -x & 0.125L & 0 \end{vmatrix} = \frac{0.125L\,dx\,\overline{a_z}}{\sqrt{x^2+(0.125L)^2}}$$

$$d\overline{H} = \frac{I\,d\overline{L}\times\overline{a_R}}{4\pi R^2} = \frac{I\ 0.125L\,dx\,\overline{a_z}}{4\pi\left[x^2+0.125L\right]^{3/2}}$$

$$\therefore \overline{H} = \int_{x=-L/8}^{L/8} d\overline{H} = 2\int_{x=0}^{L/8} d\overline{H} = \frac{2I\times 0.125L}{4\pi}\int_{x=0}^{L/8}\frac{dx\,\overline{a_z}}{\left[x^2+(0.125L)^2\right]^{3/2}}$$

put $x = 0.125L\tan\theta$; $dx = 0.125\,L\sec^2\theta$.
for $x=0$, $\theta_1 = 0^\circ$, and $x = \frac{L}{8}$, $\theta_2 = 45^\circ$

$$\therefore \overline{H} = \frac{2I\times 0.125L}{4\pi}\int_{\theta=0}^{45}\frac{0.125L\sec^2\theta\,d\theta}{\cancel{4\pi}(0.125L)^3\sec^3\theta}\overline{a_z}$$

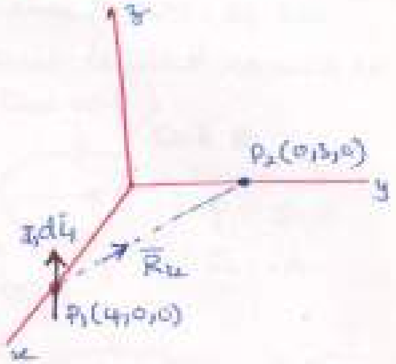
$$= \frac{2I}{4\pi(0.125L)}\int_{\theta=1}^{45}\cos\theta\,d\theta\,\overline{a_z} = \frac{1.2732\,I}{L}\left[\sin\theta\right]_0^{45}\overline{a_z}$$

$$= \frac{0.9I}{L}\overline{a_z}$$

$\therefore$ All the sides produce H in the same direction. Hence $\overline{H}$ at centre is

$$\overline{H}_{total} = 4\overline{H} = \frac{3.6I}{L}\overline{a_z}$$

// Find the incremental field strength at P_2 due to the current element of $2\pi \bar{a}_z$ μAm at P_1. The coordinates are $P_1(4,0,0)$ & $P_2(0,3,0)$



According to Biot-Savart law,

$$d\bar{H}_2 = \frac{I_1 d\bar{L}_1 \times \bar{a}_{R_{12}}}{4\pi R_{12}^2}$$

$$\bar{R}_{12} = (0-4)\bar{a}_x + (3-0)\bar{a}_y + 0\bar{a}_z$$

$$\therefore \bar{a}_{R_{12}} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{-4a_x + 3a_y}{\sqrt{16+9}} = \frac{-4\bar{a}_x + 3\bar{a}_y}{5}$$

while $I_1 d\bar{L}_1 = 2\pi \bar{a}_z$ μAm

$$\therefore I_1 d\bar{L}_1 \times \bar{a}_{R_{12}} = -\frac{4}{5}\times 2\pi \bar{a}_y - \frac{3}{5} 2\pi \bar{a}_x = -\frac{2\pi}{5}\left[3a_x + 4\bar{a}_y\right]$$

$$\therefore dH_2 = \frac{-\frac{2\pi}{5}\left[3\bar{a}_x + 4\bar{a}_y\right]}{4\pi(5^2)} = -4\times 10^{-3}\left[3\bar{a}_x + 4\bar{a}_y\right] \text{ μA/m.}$$

$$\boxed{dH_2 = -12\bar{a}_x - 16\bar{a}_y \text{ nA/m}}$$

$\nabla \times H$ in various coordinate systems

$$\nabla \times H = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \begin{matrix} \\ \leftarrow \nabla \\ \leftarrow H \end{matrix} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\bar{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\bar{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\bar{a}_z$$

In Cartesian coordinates

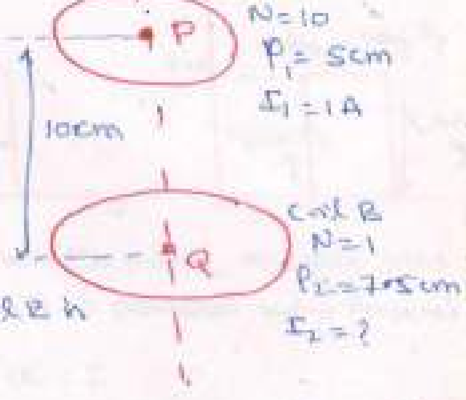
$$\nabla \times H = \frac{1}{\rho}\begin{vmatrix} \bar{a}_\rho & \rho\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_\rho & \rho H_\phi & H_z \end{vmatrix} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\bar{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\bar{a}_\phi + \left(\frac{1}{\rho}\frac{\partial (H_\phi \rho)}{\partial x} - \frac{\partial H_\rho}{\rho\, \partial \phi}\right]\bar{a}_z$$

Cylindrical coordinates

$$\nabla \times H = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\, \bar{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & r H_\theta & r\sin\theta\, H_\phi \end{vmatrix}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial H_\phi \sin\theta}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right]\bar{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial r H_\phi}{\partial r}\right]\bar{a}_\theta + \frac{1}{r}\left[\frac{\partial (r H_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right]\bar{a}_\phi$$

Two narrow circular coils A & B have common axis and are placed 10 cm apart. The coil has 10 turns of radius 5cm with a current of 1A passing through it. The coil B has a single turn of radius 7.5cm. If the magnetic field at the centre of coil A is to be zero, what current should be passed through coil B. (May 2005, Nov 2005, Nov 2009, Dec 2011)



$\overline{H}$ at the centre of the circular coil with N turns is given by

$$|\overline{H}| = \frac{NI}{2\rho_1} \qquad \rho_1 = \rho_1 = 5\,cm$$

$$= \frac{10 \times 1}{2 \times 5 \times 10^{-2}} = 100\ A/m$$

The $\overline{H}$ at the centre of coil A ie point P due to coil B is

$$|\overline{H}| = \frac{I\rho_2^2}{2(\rho_2^2+z^2)^{3/2}}$$

where $\rho = \rho_2 = 7.5cm$
$I = I_2$
z = distance between point P & coil
= 10cm

$$\therefore |\overline{H}| = \frac{I_2(7.5\times10^{-2})^2}{2\left[(7.5\times10^{-2})^2 \times (10\times10^{-2})^2\right]^{3/2}}$$

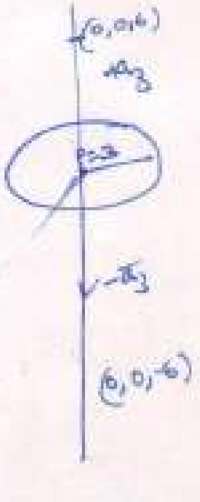
$$= 1.44 I_2$$

The total $\overline{H}$ at $P = 100 + 1.44 I_2$ which must be zero

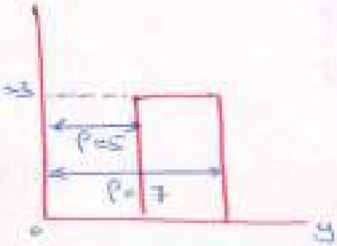
$$\therefore \quad 100 + 1.44 I_2 = 0$$

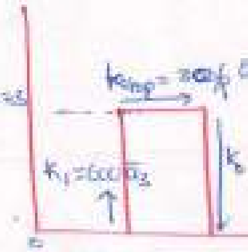
$$I_2 = -\frac{100}{1.44} = -69.44\,A$$

The –ve sign indicates that direction of current I_2 is opposite to the direction of current I_1



The region is a free space enclosed by planes $z=0$, and $z=3$ cm and by cylinders, $\rho=5$ cm & $\rho=7$ cm, forms a toroid with a rectangular cross section. A surface current $\bar{K}=600\bar{a}_z$ A/m flows on the inner surface. (a) Determine the current on the other three surface. (b) Find $\bar{H}$ within the toroid (c) find the total flux within the toroid.





The linear current density $\bar{K}=600\bar{a}_z$ A/m on the inner surface $\rho=5$ cm

The total current on the inner surface

$$I=|\bar{K}_i|\times 2\pi\rho = 600\times 2\pi\times 0.05 = 60\pi \text{ A}$$

(a) The same current flows on the outer surface towards negative z-axis in the toroid having $\rho=7$ cm

Thus the linear current density on the outer surface is in negative z-direction

$$\therefore \bar{K}_o = -\frac{60\pi}{2\pi\rho_o}\bar{a}_z = \frac{-60\pi}{2\pi\times 0.07}\bar{a}_z = -428.57\bar{a}_z \text{ A/m}$$

At $z=3$cm the near current density is dependent on the radius ρ, the linear current density at $z=0.03$m surface is

$$\bar{K}_{top}' = |\bar{K}_i|\times\frac{2\pi\times 0.05}{2\pi\rho}\bar{a}_\rho = |\bar{K}_i|\times\frac{0.05}{\rho}\bar{a}_\rho$$

$$= \frac{30}{\rho}\bar{a}_\rho \text{ A/m}$$

Similarly at $z=0.0$ surface

$$\bar{K}_{bottom} = -|\bar{K}_i|\times\frac{0.05}{\rho}\bar{a}_\rho = -\frac{30}{\rho}\bar{a}_\rho \text{ A/m.}$$

(b) within the toroid, magnetic field intensity is obtained using Amper Law

$$\oint \bar{H}\cdot d\bar{l} = I$$

$$2\pi\rho H_\phi = 60\pi \Rightarrow H_\phi = \frac{30}{\rho}\text{ A/m} \text{ (or}$$

$$H_\phi = \frac{30}{\rho}\bar{a}_\phi \text{ A/}$$

(c) Total flux within the toroid is obtained by integrating of flux density in the region of $0<\rho<5$ & $0<z<3$cm

The flux density inside the toroid

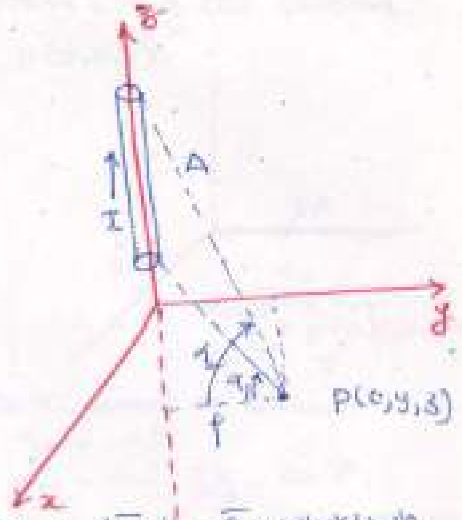
$$B = \mu_0 H = \frac{120\pi\times 10^{-7}}{\rho}\times\bar{a}_\phi$$

$$\text{Total flux } \phi = \int_0^{0.03}\int_{0.05}^{0.07}|\bar{B}|\,d\rho\,dz$$

$$= \int_0^{0.03}\int_{0.05}^{0.07}\frac{30}{\rho}\mu_0\,d\rho\,dz$$

$$= 12\pi\times 0.03\times\ln\left(\frac{7}{5}\right)\times 10^{-6} = 0.381\ \mu\text{ Web}$$

For the finite-length current element on the z-axis, as shown in figure, use the Biot-Savart law to derive $\overline{H}_\phi = \frac{I}{4\pi\rho}[\sin\alpha_2 - \sin\alpha_1]\overline{a}_\phi$.



Let a differential element at point A(0,0,z') be

$$Id\overline{L} = I\,dz'$$

To find $\overline{H}$ at B,

The vector from A to B

$\overline{R}_{AB} = \overline{B}-\overline{A} = (y\overline{a}_y + z\overline{a}_z) - (z'\overline{a}_z) = y\overline{a}_y + (z-z')\overline{a}_z$

Its magnitude is

$|\overline{R}_{AB}| = \sqrt{y^2+(z-z')^2}$

According to Biot-Savart's law

$$d\overline{H} = \frac{Id\overline{L}\times\overline{R}_{AB}}{4\pi|\overline{R}_{AB}|^3} \text{ A/m}$$

$$= \frac{I(\overline{a}_z\,dz')\times(y\overline{a}_y+(z-z')\overline{a}_z)}{4\pi|y^2+(z-z')^2|^{3/2}} = \cancel{\frac{I}{4\pi}}$$

$$= \frac{I}{4\pi|y^2+(z-z')^2|^{3/2}}\begin{vmatrix} \overline{a}_x & \overline{a}_y & \overline{a}_z \\ 0 & 0 & dz \\ 0 & y & z-z' \end{vmatrix}$$

$$= \frac{Iy\,dz'\,\overline{a}_x}{4\pi[y^2+(z-z')^2]^{3/2}}$$

The direction of $d\overline{H}$ is $-\overline{a}_x$ which is $\overline{a}_\phi$ at point B. In cylindrical coordinate system using $y=\rho$ at point B

$$d\overline{H}_\phi = \frac{I\rho\,dz'}{4\pi[\rho^2+(z-z')^2]^{3/2}}$$

Assuming $\tan\theta = \frac{z-z'}{\rho} \Rightarrow \sec^2\theta\,d\theta = \frac{dz'}{\rho}$

$$\therefore H_\phi = \frac{I\rho}{4\pi}\int_{\alpha_1}^{\alpha_2}\frac{\rho\sec^2\theta\,d\theta}{[\rho^2+\rho^2\tan^2\theta]^{3/2}}$$

$$= \frac{I}{4\pi\rho}\int_{\alpha_1}^{\alpha_2}\frac{\sec^2\theta\,d\theta}{\sec^3\theta} = \frac{I}{4\pi\rho}\int_{\alpha_1}^{\alpha_2}\cos\theta\,d\theta$$

$$H_\phi = \frac{I}{4\pi\rho}\sin\theta\Big|_{\alpha_1}^{\alpha_2}$$

$$\boxed{\therefore \overline{H} = \frac{I}{4\pi\rho}[\sin\alpha_2 - \sin\alpha_1]\overline{a}_\phi}$$

Hence proved.

In cylindrical coordinate system a magnetic field is given as $\overline{H} = (2\rho-\rho^2)a_\phi$ A/m $0\le\rho\le1$. (a) Determine the current density as a function of ρ within this cylinder. (b) What total current passes through the surface $z=0$, $0\le\rho\le1$, in the $\overline{a}_z$ direction?

Given magnetic field $\overline{H} = (2\rho-\rho^2)\overline{a}_\phi$ A/m $0\le\rho\le1$

(a) The current density

$$\overline{J} = \nabla\times\overline{H} = -\frac{\partial H_\phi}{\partial z}\overline{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial\rho}\overline{a}_z$$

$$= 0.0\overline{a}_\rho + 0.0\overline{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho(2\rho-\rho^2))}{\partial\rho}\overline{a}_z$$

$$= (4-3\rho)\overline{a}_z$$

(b) Total current passing through the surface $z=0$, $0\le\rho\le1$ in $\overline{a}_z$ direction

$$I = \int_0^{2\pi}\int_0^1 J_z\,\rho\,d\rho\,d\phi = \int_0^{2\pi}\int_0^1(4-3\rho)\rho\,d\rho\,d\phi$$

$$= \int_0^{2\pi}\left(4\frac{\rho^2}{2} - 3\frac{\rho^3}{3}\right)_0^1 d\phi = 2\pi(2-1) = 2\pi = \underline{6.28\text{ A}}$$

$\therefore d\overline{S} = \rho\,d\rho\,d\phi\,\overline{a}_z$

Alternatively

using Stoke's Theorem

$$\oint\overline{H}\cdot d\overline{L} = \int(2\rho-\rho^2)\overline{a}_\phi\cdot\rho\,d\phi\,\overline{a}_\phi$$

$$= 2\pi(2-1)$$

$$= \underline{6.28\text{ A}}$$

$$\oint H\cdot dL = \int_S(\nabla\times H)\cdot d\overline{S}$$

$\overline{J}\cdot d\overline{S} =$

$(4-3\rho)\rho\,d\rho\,d\phi\,\overline{a}_z\cdot\overline{a}_z$

$= (4-3\rho)\rho\,d\rho\,d\phi$

A filamentary current of 10A is directed in from infinity to the origin on the positive x-axis, and then back out to infinity along the positive y-axis. Use the Biot-Savart's Law to find $\overline{H}$ at P(0,0,1)

Sol:- We know that Biot-Savart's law is

$$d\overline{H} = \frac{I d\overline{L} \times \overline{a}_R}{4\pi R^2} = \frac{I d\overline{L} \times \overline{R}}{4\pi R^3} \text{ A/m}$$

Initially considering x-axis, vector

$$\overline{R} = -x\overline{a}_x + \overline{a}_z$$

and $I d\overline{L} = 10\, dx\, \overline{a}_x$

$$\therefore \overline{H} = \frac{10}{4\pi}\int_{\infty}^{\infty} \frac{dx(\overline{a}_x + 0\times\overline{a}_y + 0\times\overline{a}_z)\times(-x\overline{a}_x + 0\times\overline{a}_y + \overline{a}_z)}{(1+x^2)^{3/2}}$$

$$= \frac{10}{4\pi}\int_{\infty}^{0} \frac{dx\ \overline{a}_y}{(1+x^2)^{3/2}} = \frac{10}{4\pi} \frac{x}{\sqrt{1+x^2}}\Bigg|_{\infty}^{0} a_y = \frac{10}{4\pi}\overline{a}_y$$

Similarly for y-axis

$$\overline{H} = \frac{10}{4\pi}\int_{0}^{\infty} \frac{dy(0\times\overline{a}_x + \overline{a}_y + 0\times\overline{a}_z)\times(0\times\overline{a}_x - y\overline{a}_y + \overline{a}_z)}{(1+y^2)^{3/2}}$$

$$= \frac{10}{4\pi}\int_{0}^{\infty} \frac{dy}{(1+y^2)^{3/2}}\overline{a}_x = \frac{10}{4\pi}\left| \frac{y}{\sqrt{1+y^2}} \right|_{0}^{\infty} \overline{a}_x = \frac{10}{4\pi}\overline{a}_x$$

The effective MFI is given by

$$\overline{H} = \frac{10}{4\pi}(\overline{a}_x + \overline{a}_y) = 0.796\overline{a}_x + 0.796\overline{a}_y \text{ A/m}$$

For the current on x-axis.

$\alpha_{1x} = -90°$

$\alpha_{2x} = \tan^{-1}\frac{0.4}{0.3} = 53.1°$

$\rho = 0.3$ measured from x-axis.

$$\therefore \overline{H}_{2(x)} = \frac{8}{4\pi(0.3)}(\sin 53.1° + 1)a_\phi$$

$$= \frac{12}{\pi}\overline{a}_\phi$$

The unit vector must also be referred to the x-axis. $(-\overline{a}_y)$

$$\therefore H_{2(x)} = -\frac{12}{\pi}a_\phi$$

For the current on y-axis

$\alpha_{1y} = -\tan^{-1}\left(\frac{0.3}{0.4}\right) = -36.9°$ $\alpha_{2y} = 90°$

$\rho_y = 0.4$

$$\therefore \overline{H}_{2y} = \frac{8}{4\pi(0.4)}(1+\sin 36.9°)(-\overline{a}_z)$$

$$= -\frac{8}{\pi}\overline{a}_z \text{ A/}$$

$$\therefore \overline{H} = \overline{H}_{2x} + \overline{H}_{2y} = \frac{-20}{\pi}\overline{a}_z$$

$$= -6.37\overline{a}_z \text{ A/m}$$

Find the magnetic field intensity at point P for the circuit shown in the fig.

Section I

From the figure $\alpha_1 = \tan^{-1}\frac{5}{20} = 14.036°$

$\alpha_2 = \tan^{-1}\frac{5}{20} = 14.036$

but $\alpha_1 = -14.036$

α_1 is -ve as point A is below P.

If the loop is placed in xy plane, direction of $\overline{H}$ at P is going into the paper normal to xy plane according to right hand Thumb rule

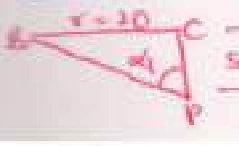
This is $-\overline{a}_z$ direction.

$$\therefore \overline{H}_1 = \frac{I}{4\pi r}[\sin\alpha_2 - \sin\alpha_1](-\overline{a}_z)$$

$$= \frac{10}{4\pi\times 20}[\sin 14.036 - \sin(-14.036)][\overline{a}_z]$$

$$= -0.0193\,\overline{a}_z \text{ A/m}$$

Section II



As the point C & P are colinear

$\alpha_2 = 0$

5m $\alpha_1 = \tan^{-1}\frac{20}{5} = 75.96$

$\alpha_1 = -75.96$

and according to RHT Rule $-\overline{a}_z$

$$H_2 = \frac{1}{4\pi r}[\sin 0 - \sin(-75.96)]\overline{a}_z$$

$$= -0.1544\,\overline{a}_z \text{ A/m}$$

Section III COE

$\overline{H}$ at the centre of the circular loop

$$\overline{H} = \frac{I}{2R}\overline{a}_N$$

$$\therefore \overline{H}_3 = \frac{\left(\frac{I}{2R}\right)}{2}(-\overline{a}_z) = \frac{10}{4\times 5}(-\overline{a}_z) = -0.5\overline{a}_z$$

A/m

Section IV

$H_4 = H_2 = -0.1544\,\overline{a}_z$ A/m

$$\therefore \overline{H} = \overline{H}_1 + \overline{H}_2 + \overline{H}_3 + \overline{H}_4$$

$$= -0.8281\,\overline{a}_z \text{ A/m}$$

UNIT-VI : FORCE IN MAGNETIC FIELD

There are at least three ways in which force due to magnetic field can be experienced. The force can be

(a) due to a moving charged particle in a field $\overline{B}$

(b) on a current element in an external $\overline{B}$ field, or

(c) between two current elements.

A. Force on a Charged Particle

In an electric field, the electric force $\overline{F_e}$ on a stationary or moving electric charge Q in an electric field is given by Coulomb's law and is related to the electric field intensity $\overline{E}$ as

$$\overline{F_e} = Q\overline{E} \longrightarrow ①$$

$\overline{F_e}$ & $\overline{E}$ have the same direction (for a +ve charge)

A magnetic field can exert force only on a moving charge. From experiments, it is found that the magnetic force $\overline{F_m}$ experienced by a charge Q moving with a velocity $\overline{u}$ in a magnetic field $\overline{B}$ is

$$\overline{F_m} = Q\overline{u} \times \overline{B} \longrightarrow ②$$

It is clear that $\overline{F_m}$ is perpendicular to both $\overline{u}$ & $\overline{B}$

From equations ① & ②

- * $\overline{F_e}$ is independent of velocity of the charge and can perform work on the charge and change its KE (Kinetic Energy)
- * $\overline{F_m}$ depends on the charge velocity and is normal to it.
- * However, $\overline{F_m}$ cannot perform work because it is at right angle to the direction of motion of the charge ($\overline{F_m} \cdot d\overline{l}) = 0$; it does not cause an increase in KE of the charge.
- * The magnitude of $\overline{F_m}$ is generally small in comparison to $\overline{F_e}$ except at high velocities.

For a moving charge Q in the presence of both electric and magnetic fields, the total force on the charge is given by

$$\overline{F} = \overline{F_e} + \overline{F_m}$$

$$\text{or} \quad \boxed{\overline{F} = Q\left[\overline{E} + \overline{u} \times \overline{B}\right]}$$

— Lorentz Force Equation
It relates mechanical force to electrical force.

If the mass of the charged particle moving in $\overline{E}$ & $\overline{B}$ fields is m, by Newton's second law of motion $\overline{F} = m\frac{d\overline{u}}{dt} = Q(\overline{E} + \overline{u} \times \overline{B})$.

Note: * Energy can be transferred only by means of the electric field.

B. Force on a Current Element

The force on a charged particle moving through a steady magnetic field may be written as the differential force exerted on a differential element of charge,

$$d\bar{F} = dQ\,\bar{u}\times\bar{B} \longrightarrow ①$$

Physically, the differential element of charge consists of a large number of very small discrete charges occupying a volume which, although small, is much larger than the average separation between the charges. If we ~~say that~~ consider an element of moving charge in an electron beam, the force is merely the sum of forces on the individual electrons in that small volume element, but if we are considering an element of moving charge within a conductor, the total force is applied to a solid conductor itself. So let our attention to the forces in current-carrying conduct

To determine the force on a current element $Id\bar{L}$ of a current-carryi conductor due to magnetic field $\bar{B}$, let us use convection current density in term of velocity of the volume charge density

$$\bar{J} = \rho_v\bar{u} \longrightarrow ②$$

$$\bar{J} = \frac{I}{A} = \frac{dQ}{dt}\times\frac{1}{dz\,dy}\times\frac{dx}{dx}$$
$$= \frac{dQ}{dx\,dy\,dz}\times\frac{dx}{dt} = \rho_v\bar{u}_x$$
$$\bar{u} = \frac{dx}{dt}$$

The differential charge dQ may also be expressed in terms of volume charge density $dQ = \rho_v dv$

$\therefore$ eq ① can be written as

$$\therefore d\bar{F} = \rho_v dv\,\bar{u}\times\bar{B} = \underline{\rho\bar{u}}\times\bar{B}\,dv$$

$$\text{or } d\bar{F} = \bar{J}\times\bar{B}\,dv \longrightarrow ③ \quad \therefore \bar{F} = \int_v (\bar{J}\times\bar{B})\,dv$$

recalling the relationship between current elements

$$Id\bar{L} = \bar{K}ds = \bar{J}dv \rightarrow$$

line charge ~~volume~~ surface volume charge densities

$\therefore$ for a differential current element (filament)

$$d\bar{F} = Id\bar{L}\times\bar{B} \longrightarrow ④$$

and $$d\bar{F} = \bar{K}\times\bar{B}\,ds \longrightarrow ⑤$$ Surface current density

$\therefore$ Total force is given by

$$F = \int_{vol} d\bar{F} = \int_{vol} \bar{J}\times\bar{B}\,dv$$
$$= \int_s \bar{K}\times\bar{B}\,ds$$
$$\text{or } \boxed{\bar{F} = \oint I d\bar{L}\times\bar{B}} = -I\oint\bar{B}\times d\bar{L}$$
$$\text{or } F = I\bar{L}\times\bar{B}$$
$$= BIL\sin\theta$$

The Magnetic field $\bar{B}$ is defined as the force per unit current element.

$\because Id\bar{L} = \bar{J}dv$

$$\therefore F = \int_v (\bar{J}\times\bar{B})\,dv$$
$$= \int_v \bar{J}dv\times\bar{B}$$
$$= \int I d\bar{L}\times B$$
$$= I\int dL\times B \longrightarrow$$

Consider two infinitely long parallel conductors carrying currents I_1 and I_2 respectively in the same direction. Let the distance between the conductors be d meters. Since I_1 and I_2 are in the same direction, an attractive force exists between the two conductors.

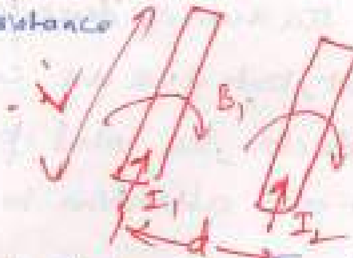
We know that magnetic field intensity due to current carrying conductor at a distance is given by

$$\overline{H} = \frac{I}{2\pi\rho}\,\overline{a}_\phi \longrightarrow ①$$

∴ the magnetic field intensity due to I_1 on the second conductor at a distance d is

$$\overline{H}_1 = \frac{I_1}{2\pi d}\,\overline{a}_\phi \quad A/m$$

$$\text{and } \overline{B}_1 = \frac{\mu I_1}{2\pi d}\,\overline{a}_\phi \quad wb/m^2$$



diagram

Let a current element on second conductor be $I_2\,d\overline{L}$. The differential force on the second conductor is

$$d\overline{F}_2 = I_2\,d\overline{L} \times \overline{B}_1$$

Since $d\overline{L}$ is normal to B_1,

$$d\overline{F}_2 = I_2\,d\overline{L}\,B_1\,\overline{a}_n = I_2\,dL\,\frac{\mu I_1}{2\pi d}\,\overline{a}_n$$

Integrating $\quad F_2 = \int dF_2 = \frac{\mu I_1 I_2}{2\pi d}\,\overline{a}_n \int d\overline{L} = \frac{\mu I_1 I_2 L}{2\pi d}\,\overline{a}_n$

Then force per unit length $\quad F_2 = \frac{\mu I_1 I_2}{2\pi d}\,\overline{a}_n$

Similarly, the force per unit length due to current I_2 in the first conductor is

$$F_1 = \frac{\mu I_1 I_2}{2\pi d}\,\overline{a}_n$$

$$\text{Then } |\overline{F_1}| = |\overline{F_2}|$$

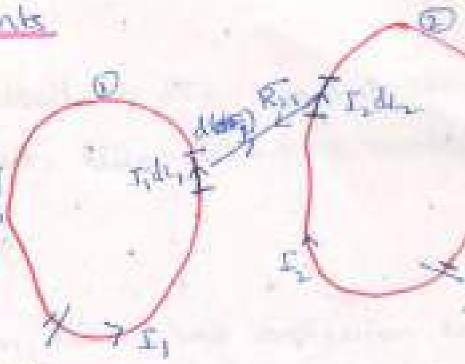
The force between the two conductors is attractive.

Note:- If currents I_1 and I_2 are in opposite directions

$$\overline{F_1} = -\overline{F_2}$$

ie Force acts in opposite directions and is repulsive.

C. Force between Two Current Elements



Let us now consider the force between two elements $I_1 d\bar{L}_1$ and $I_2 d\bar{L}_2$. According to Biot-Savart's law, both current elements produce magnetic fields. So we may find the force $d(d\bar{F}_1)$ on element $I_1 d\bar{L}_1$ due to the field $d\bar{B}_2$ produced by element $I_2 d\bar{L}_2$ as shown in figure.

$$H = \frac{I dL \times a_R}{4\pi R^2}$$

$$\therefore d(d\bar{F}_1) = I_1 d\bar{L}_1 \times d\bar{B}_2$$

$B = \mu H$

But from Biot-Savart's law

$$d\bar{B}_2 = \frac{\mu_0 I_2 d\bar{L}_2 \times \bar{a}_{R_{21}}}{4\pi R_{21}^2}$$

Hence

$$d(d\bar{F}_1) = \frac{\mu_0 I_1 d\bar{L}_1 \times (I_2 d\bar{L}_2 \times \bar{a}_{R_{12}})}{4\pi R_{21}^2}$$

This equation is essentially the law of force between two current elements and is analogous to Coulomb's law, which expresses the force between two stationary charges. From above equation, we obtain the total force $\bar{F}_1$ on current loop 1 due to current loop 2 is

$$\bar{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\bar{L}_1 \times (d\bar{L}_2 \times \bar{a}_{R_{21}})}{R_{21}^2} = -\bar{F}_2$$

$\boxed{\bar{F}_1 = -\bar{F}_2}$ Obeys Newton's Third Law

Force between Two Straight, Long & Parallel Conductors Carrying Current

$$\vec{F} = I\bar{L} \times \bar{B} = |B||I||L| \sin\theta = BIL \sin\theta \rightarrow F_2 = B_1 I_2 L \sin\theta \quad \sin\theta = 1 \quad \theta = 90^\circ$$

B & I are Norm to each other

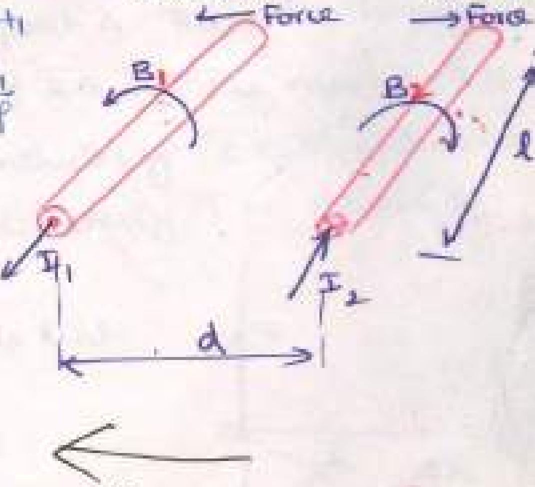
$= B_1 I_2 L$

Force exerted on a conductor is given by $F = B_1 I_2 L$

$B_1 = \mu_0 H_1 = \frac{\mu_0 I_1}{2\pi\rho}$

$\rho = d.$

$$\boxed{F = \frac{\mu_0 I_1 I_2 l}{2\pi d} N}$$



If the directions of currents through the conductors are same, then two conductors attract each other. While if the directions of the currents through the conductors are opposite, then two conductors repel each other.

MAGNETIC TORQUE & MOMENT

The general expression for the force exerted on the current element, If we consider a filamentary closed circuit, then the force exerted is

$$\overline{F} = \oint I d\overline{L} \times \overline{B} = -I \oint \overline{B} \times d\overline{L}$$

If we assume magnetic flux density uniform, then above expression can be modified as

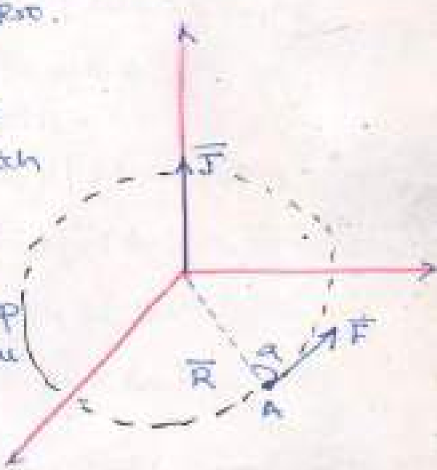
$$\overline{F} = -I\,\overline{B} \times \oint d\overline{L}$$

But we know that for a closed circuit $\boxed{\oint d\overline{L} = 0}$? And thus the force on a closed filamentary circuit is zero in the uniform magnetic field.

If the field is not uniform, then force on closed circuit is not zero.

This is valid for circuits with surface current or volume current density. Even though the current is divided into the filaments, the force on each filament will be zero and hence the effective total force will be zero. Hence in an uniform magnetic field, the force experienced by a closed circuit carrying current is always equal to zero.

Let us consider a new vector quantity which is not equal to zero, even if the force is zero; which is called magnetic torque or moment of force.



Def - The torque $\overline{T}$ (or moment of force) on the loop is the vector product of the force $\overline{F}$ and the moment arm $\overline{R}$ measured in (N-m)

Torque about the origin

$$\boxed{\overline{T} = \overline{R} \times \overline{F} \text{ N m}}$$

Consider a point A at which force $\overline{F}$ is applied as shown in fig. let $\overline{R}$ be the arm from the origin O to point A. Then the torque $\overline{T}$ about the origin is a vector product of $\overline{R}$ & $\overline{F}$. The magnitude of Torque is equal to the product of magnitudes of $\overline{R}$ & $\overline{F}$ and sine of the angle between $\overline{R}$ & $\overline{F}$, while the direction of $\overline{T}$ is normal to both $\overline{R}$ & $\overline{F}$

Now consider the two forces $\overline{F_1}$ & $\overline{F_2}$ such that

$\overline{F_1} = -\overline{F_2}$

$\therefore \overline{T} = \overline{R_1} \times \overline{F_1} + \overline{R_2} \times \overline{F_2}$

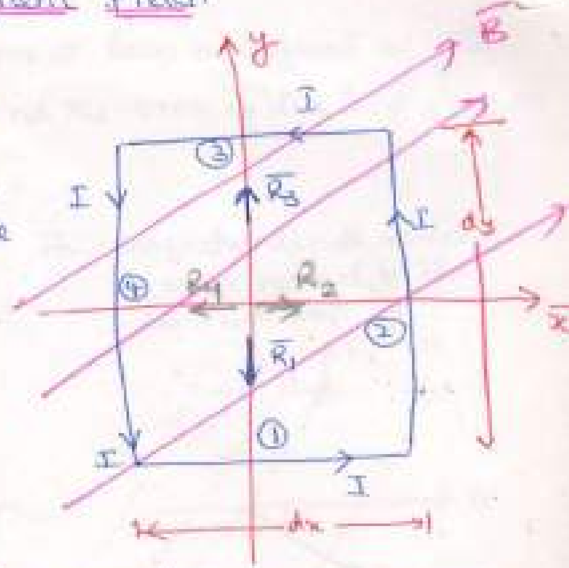
$= (R_1 - R_2) \times \overline{F_1} \quad \because \overline{F_2} = -\overline{F_1}$

$\overline{T} = \overline{R_{21}} \times \overline{F_1}$

It is clear that when total force is zero the torque is independent of the choice of the origin.

Torque on a Current Loop Placed in Magnetic Field.

Consider a differential current loop placed in x-y plane in the magnetic field $\bar{B}$. The loop is placed in the plane such that the sides of the loop are parallel to the axes respectively. Let dx and dy be the lengths of the sides of the loop as shown in the figure.



Assume that the current in the loop flows in anticlockwise direction. Let the magnetic field at the centre of the loop be $\bar{B}_0$ and is uniform at all points. As per the concept the total force on the closed loop is zero. Assume that the origin for the torque is at the centre of the loop.

Consider side 1 of the differential loop. The differential force exerted on side 1 is given by

$$d\bar{F}_1 = I d\bar{L}_1 \times \bar{B}_0 = I dx\, \bar{a}_x \times \bar{B}_0$$

Let $\bar{B}_0 = B_{0x}\bar{a}_x + B_{0y}\bar{a}_y + B_{0z}\bar{a}_z$

$$\therefore d\bar{F}_1 = I dx\, \bar{a}_x \times (B_{0x}\bar{a}_x + B_{0y}\bar{a}_y + B_{0z}\bar{a}_z)$$
$$= I dx\, (B_{0y}\bar{a}_z - B_{0z}\bar{a}_y)$$

The moment of arm for this side is vector which extends from origin to the mid-point of the side and it is given by

$$\bar{R}_1 = -\frac{1}{2} dy\, \bar{a}_y$$

∴ The torque on side 1 is given by

$$d\bar{T}_1 = \bar{R}_1 \times d\bar{F}_1$$
$$= -\frac{1}{2} dy\, \bar{a}_y \times I dx (B_{0y}\bar{a}_z - B_{0z}\bar{a}_y)$$
$$= -\frac{1}{2} I dx dy\, (B_{0y}\, \bar{a}_x)$$

For side 3 of the loop, the differential force exerted is given by

$$d\bar{F}_3 = I d\bar{L}_3 \times \bar{B}_0 = -I(dx\,\bar{a}_x) \times (B_{0x}\bar{a}_x + B_{0y}\bar{a}_y + B_{0z}\bar{a}_z)$$
$$d\bar{F}_3 = -I dx\, (B_{0y}\bar{a}_z - B_{0z}\bar{a}_y)$$

The moment arm R_3 extends from origin to the mid point of the side is given by

$$\bar{R}_3 = \frac{1}{2} dy\, \bar{a}_y$$

Thus the torque on side 3 is given by

$$d\bar{T}_3 = \bar{R}_3 \times d\bar{F}_3$$
$$= \frac{1}{2} dy\, \bar{a}_y \times [-I dx (B_{0y}\bar{a}_z - B_{0z}\bar{a}_y$$
$$= -\frac{1}{2} I dx dy\, B_{0y}\bar{a}_x$$

Similarly for side 2.

$$d\bar{T}_2 = \bar{R}_2 \times d\bar{F}_2$$
$$= \left(\frac{1}{2} dx\, \bar{a}_x\right) \times [I dy\, a_y \times (B_{0x}\bar{a}_x + B_{0y}\bar{a}_y + B_{0z}\bar{a}_z$$
$$= \frac{1}{2} dx\, \bar{a}_x \times [-I dy (B_{0x}\bar{a}_z - B_{0z}\bar{a}_x)]$$
$$= -\frac{1}{2} dx dy\, I (-B_{0x}\bar{a}_y)$$
$$= +\frac{1}{2} dx dy\, I B_{0x}\bar{a}_y$$

for side 4

$$d\bar{T}_4 = \bar{R}_4 \times d\bar{F}_4 = d\bar{T}_2$$
$$= \frac{1}{2} dx dy\, I B_{0x}\bar{a}_y$$

∴ Total torque is given by

$$d\bar{T} = d\bar{T}_1 + d\bar{T}_2 + d\bar{T}_3 + d\bar{T}_4$$
$$= I dx\, dy\, (B_{0x}\bar{a}_y - B_{0y}\bar{a}_x)$$

But $(B_{0x}\bar{a}_y - B_{0y}\bar{a}_x) = \bar{a}_z \times (B_{0x}\bar{a}_x + B_{0y}\bar{a}_y + B_{0z}\bar{a}_z)$

$$\therefore d\bar{T} = I dx dy\, [\bar{a}_z \times \bar{B}_0]$$
$$= I dx dy\, \bar{a}_z \times \bar{B}_0$$
$$d\bar{T} = I d\bar{S} \times \bar{B}_0$$

Since $\bar{B}_0$ is same everywhere

$$\boxed{\therefore d\bar{T} = I d\bar{S} \times \bar{B}_0}$$

Magnetic Dipole Moment

Def:- The magnetic dipole moment of a current loop is defined as the product of current through the loop and the area of the loop, directed normal to the current loop.

$$\therefore \quad \overline{m} = (IS)\,\overline{a}_N \ \mathrm{A\,m^2}$$

$\overline{m}$ - magnetic dipole moment a vector normal to the current loop.

Now the torque equation

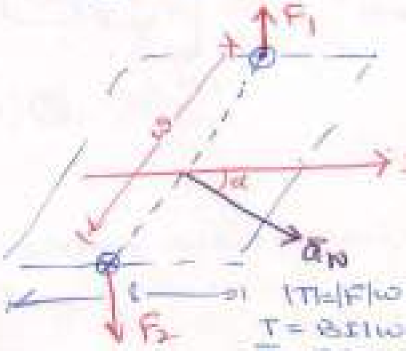
$$d\overline{T} = I d\overline{S} \times \overline{B}$$

integrating the above equation

$$\overline{T} = I\,\overline{S} \times \overline{B}$$

$$= IS\,\overline{a}_N \times \overline{B}$$

$$\boxed{\overline{T} = \overline{m} \times \overline{B}}$$

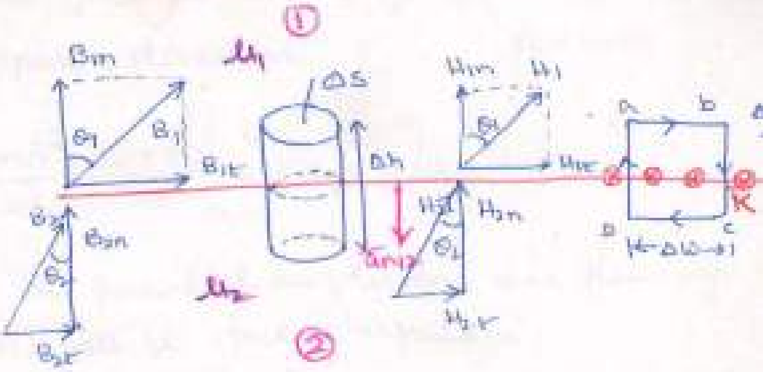


$|T| = |F|w\sin\alpha$
$T = BIlw\sin\alpha$
$\overline{T} = BIS\sin\alpha\,\overline{a}_N$
$T = I\overline{S} \times \overline{B}$

Although this expression was obtained by using a rectangular loop it is generally applicable in determining the torque on a planar loop of any arbitrary shape. The only limitation is that the magnetic field must be uniform. It should be noted that the torque is in the direction of the axis of rotation.

Magnetic Boundary Conditions

The conditions of the magnetic field existing at the boundary of the two media when the magnetic field passes from one medium to other are called **boundary conditions for magnetic fields** or **magnetic boundary conditions**. Means conditions of $\overline{B}$ & $\overline{H}$ at the boundary.



To study $\overline{B}$ & $\overline{H}$ at the boundary, both the vectors are resolved into two components ① Tangential to boundary &
② Normal to boundary.

Consider a boundary between two isotropic, homogeneous linear materials with different permeabilities μ_1 & μ_2. To determine the boundary conditions, let us use the gaussian surface & closed path.

Boundary conditions for normal component & Tangential component

To find conditions for $\overline{B}$, consider a Gaussian surface (a cylinder).

Applying Gauss's law

$$\oint \overline{B}\cdot d\overline{S} = 0$$

[illegible]

$$\Rightarrow \oint_{top} \overline{B}\cdot d\overline{S} + \oint_{bottom} \overline{B}\cdot d\overline{S} + \oint_{lateral} \overline{B}\cdot d\overline{S} = 0$$

$$= B_{N1}\oint_{top} d\overline{S} + \oint d\overline{S} + 0 \quad (\text{as } \Delta h \to 0)$$

$$B_{1N}\Delta S\,\bar{a}_n + B_{2N}\Delta S(-\bar{a}_n)$$

$$\therefore B_{1N}\Delta S - B_{2N}\Delta S = 0$$

(or) $\boxed{B_{1N} = B_{2N}}$ Normal component of $\overline{B}$ is continuous

& $\overline{B} = \mu\overline{H}$

$\therefore B_{1N} = B_{2N}$ can be written as

$\Rightarrow \mu_1 H_{N_1} = \mu_2 H_{N_2}$

$\therefore \boxed{\dfrac{H_{N1}}{H_{N2}} = \dfrac{\mu_2}{\mu_1} = \dfrac{\mu_{r2}}{\mu_{r1}}}$ Normal component of $\overline{H}$ is not continuous.

To find the tangential component of $\overline{H}$ consider a closed path abcda

Apply Ampere's law

$$\oint \overline{H}\cdot d\overline{L} = I =$$

$$= \int_a^b H\cdot dL + \int_b^c H\cdot dL + \int_c^d H\cdot dL + \int_d^a H\cdot dL$$

$$K.\Delta W = H_{1t}\cdot\Delta W + \left(H_{1N}\frac{\Delta h}{2} + H_{2N}\frac{\Delta h}{2}\right) - H_{2t}\Delta W$$

$$- \left(H_{2N}\frac{\Delta h}{2} + H_{1N}\frac{\Delta h}{2}\right)$$

as $\Delta h \to 0$

$$K.\Delta W = H_{1t}\Delta W - H_{2t}\Delta W$$

$$\boxed{K = H_{1t} - H_{2t}}$$

& $\boxed{K = \dfrac{B_{1t}}{\mu_1} - \dfrac{B_{2t}}{\mu_2}}$

in general $\boxed{\overline{K} = (\overline{H}_1 - \overline{H}_2)\times\bar{a}_{n12}}$

If the boundary is free of current K=0

$\therefore \boxed{H_{1t} = H_{2t}}$ & $\dfrac{B_{1t}}{\mu_1} = \dfrac{B_{2t}}{\mu_2}$

or $\boxed{\dfrac{B_{1t}}{B_{2t}} = \dfrac{\mu_1}{\mu_2}}$

If the fields make an angle θ with the normal to the interface

$$B_1\cos\theta_1 = B_{1N} = B_{2N} = B_2\cos\theta_2 \longrightarrow ①$$

& $$\frac{B_1}{\mu_1}\sin\theta_1 = H_{1t} = H_{2t} = \frac{B_2}{\mu_2}\sin\theta_2 \longrightarrow ②$$

$$\frac{②}{①} = \boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}}$$ law of refraction of magnetic flux lines at the boundary.

Eg! Two long parallel conductors carry 80A. If the conductors are separated by 30 mm. Find the force per meter of each conductor if the current flows is in opposite direction. Nov 2010

$$F = \frac{\mu_0 I_1 I_2}{2\pi r} = \frac{4\pi\times10^{-7}\times80\times80}{2\pi\times0.03} = 0.04\ N/m$$

As the ~~conductors~~ currents in parallel conductors are flowing in opposite direction, the force will be force of repulsion.

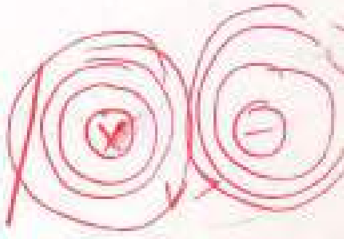
Eg! A point charge of Q = −1.2C has velocity $\bar{u} = (5\bar{a}_x + 2\bar{a}_y - 3\bar{a}_z)$ Find the magnitude of the force exerted on the charge if,

a) $\bar{E} = -18\bar{a}_x + 5\bar{a}_y - 10\bar{a}_z$ V/m

b) $\bar{B} = -4\bar{a}_x + 4\bar{a}_y + 3\bar{a}_z$ T

c) Both are present simultaneously

(sm. Nov2008)



(a) $\bar{F_e} = Q\bar{E} \Rightarrow |\bar{F_e}| = 25.4275\ N$

(b) $\bar{F_m} = Q\bar{u}\times\bar{B} \Rightarrow |F_m| = 40.1058\ N$

(c) $\bar{F} = \bar{F_e} + \bar{F_m} = Q(\bar{E} + \bar{u}\times\bar{B}) = Q\bar{E} + Q\bar{u}\times\bar{B}$

$$|\bar{F}| = 21.7329\ N.$$

Eg!- An electron with velocity $\bar{u} = (3\bar{a}_x + 12\bar{a}_y - 4\bar{a}_z)\times10^5$ m/s experience no net force at a point in a magnetic field $\bar{B} = 10\bar{a}_x + 20\bar{a}_y + 3\bar{a}_z$ mWb/m². Find E at this point. (Nov2005)

$$\bar{F} = 0 = Q(\bar{E} + \bar{u}\times\bar{B})$$

$$\therefore\ \bar{E} = -\bar{u}\times\bar{B}$$

$$= -(3\bar{a}_x + 12\bar{a}_y - 4\bar{a}_z)\times(10\bar{a}_x + 20\bar{a}_y + 3\bar{a}_z)10^5\times$$

$$= -4.4\bar{a}_x + 1.3\bar{a}_y + 11.4\bar{a}_z\ kV/m.$$

Eg!- A current of 10 A flows in each of two conducting wires parallel each other. The separation between the wires is 2 cm. Find the force per unit length of one of the wires.

~~$-60\bar{a}_z$~~ $-12\times3(+\bar{a}_x) - 4\times20(-\bar{a}_x)$

$-36 - 80$

116.

A single phase circuit comprises two parallel ~~plate~~ conductors A & B each 1cm diameter and spaced 1m apart. The conductors carry currents of +100 & -100 A respectively. Determine the magnetic field intensity at the surface of each conductor and also exactly midway between A & B.

A differential current loop has dimensions of 1m x 2m and lies in uniform field $B_0 = -0.6\bar{a}_y + 0.8\bar{a}_z$ T. The loop current is 4mA. Find torque on the loop.

$$T = I\bar{S} \times B = 4 \times 10^{-3}[1 \times 2\bar{a}_z] \times [-0.6\bar{a}_y + 0.85\bar{a}_z]$$

$$= 4.8\bar{a}_x \text{ mN.m}$$

Side 1. $F_1 = Id\bar{L}_1 \times \bar{B}_0 = 4 \times 10^{-3}(1\bar{a}_x)(-0.6\bar{a}_y + 0.85\bar{a}_z)$

$$= -3.2\bar{a}_y - 2.4\bar{a}_z \text{ mN}$$

$dL_2\ (2\bar{a}_y)$

$$F_2 = Id\bar{L}_2 \times \bar{B}_0 = 4 \times 10^{-3}(2\bar{a}_y) \times (-0.6\bar{a}_y + 0.8\bar{a}_z)$$

$$= 6.4\bar{a}_x \text{ mN.}$$

$$F_3 = 3.2\bar{a}_y + 2.4\bar{a}_z = -F$$

$$F_4 = -F_3.$$

$$\therefore T = T_1 + T_2 + T_3 + T_4 = \underset{\substack{\downarrow \\ (-\bar{a}_y)}}{R_1} \times F_1 + \underset{\substack{\downarrow \\ (0.5\bar{a}_x)}}{R_2} \times F_2 + \underset{\substack{\downarrow \\ (\bar{a}_y)}}{R_3} \times F_2 + \underset{\substack{\downarrow \\ (-0.5\bar{a}_x)}}{R_4} \times F_4.$$

$$= 4.8\bar{a}_x \text{ mN.m.}$$

The force between two long parallel conductors placed 10 cm from each other is 15 kg/m. If one conductor carries twice the current as the other, calculate the current in each conductor.

(May 2011)

$F = \frac{\mu_0 I_1 I_2}{2\pi d}$ N/m.

Let the current in conductor 1 be I then in other conductor 2 will be 2I

$\frac{15}{9.8}$

$$\boxed{15\times10^{-5}} = \frac{4\pi\times10^{-7}\, I(2I)}{(2\pi)(0.1)}$$

$$\therefore I = \sqrt{\frac{15}{40}\times10^{2}} = \underline{6.123 A}$$

(i) What is the maximum torque on a square loop of 100 turns in a field of uniform flux density 1 Wb/m²? The loop has 10cm side and carries a current of 3A. (ii) What is the magnetic moment of the loop.

(Nov 2010)

(i) max torque $\underline{T = BIS}$

for first loop $T_{max} = 1\times3\times10^{-2}$ N-m / Turn

for N turns $T_{max} = 3\times10^{-2}\times100 = 3$ N-m

(ii) magnetic Moment $m = IS~~x~~ = 3\times10^{-2}$ A-m²

If the magnetic field is $\overline{H} = (0.01/\mu_0)\,\overline{a_x}$ A/m what is the force on a charge of 1pC moving with a velocity of $10^6\,\overline{a_y}$ m/s

(Feb 2002, Nov 2006)

Force experienced by a charge Q moving with a velocity $\overline{u}$ in a magnetic field $\overline{B}$ is

given $\overline{H} = \frac{0.01}{\mu_0}\overline{a_x}$ A/m.
$Q = 1pC = 10^{-12}C$
$\overline{u} = 10^6\,\overline{a_y}$ m/s

$$\overline{F_m} = Q\,\overline{u}\times\overline{B}$$

we also know that $\overline{B} = \mu_0 H = \mu_0\left(\frac{0.01}{\mu_0}\right)\overline{a_x}$

or $\overline{B} = 0.01\,\overline{a_x}$ T

also $\overline{u}\times\overline{B} = 10^6\,\overline{a_y}\times0.01\,\overline{a_x} = -10^4\,\overline{a_z}$

Now $\overline{F_m} = 10^{-12}(\overline{u}\times\overline{B})$

$$= 10^{-12}(-10^4\,\overline{a_z}) = \underline{-10^{8}\,\overline{a_z}}\text{ N}$$

A wire 1 metre long carries a current of 10A and makes an angle of 30° with uniform magnetic field B = 1.5 wb/m². Calculate the magnitude of the force on the wire (May 2011)

Magnitude of force $F = I(\bar{L} \times \bar{B}) = ILB\sin\theta$

$= 10 \times 1.5 \times \sin 30° = 7.5N.$

A solenoid 25 cm long and of 1 cm mean diameter of the coil-turns has a uniform distributed winding of 2000 turns. If the solenoid is placed in a uniform field of 2 wb/m² flux density and current of 5A is passed through the solenoid winding, what is the maximum (i) force on the solenoid and (ii) torque on the solenoid? (May 2011)

(i) $F = I\bar{L} \times \bar{B}$

maximum force $F = ILB = 5 \times 0.25 \times 2 = 2.5$ N/turn

for N turns $F = 2.5 \times 2000 = 5000N$

(ii) Torque $T = BIS = 2 \times 5 \times 0.25\pi \times 10^{-2} = 2.5\pi \times 10^{-4}$ N-m

for N turns $T = 2.5\pi \times 10^{-4} \times 2000 = 0.5\pi$ N-m.

UNIT-V : Self & Mutual Inductance:

A wire or conductor of certain length, when twisted into coil becomes a basic inductor (as shown in figure.1). When current I passes through the coil, it produces a Magnetic field $\bar{B}$ which causes a flux ψ in each turn, there exists a self-inductance. When two such coils are placed very close to each other, there exists a mutual inductance between the two coils.

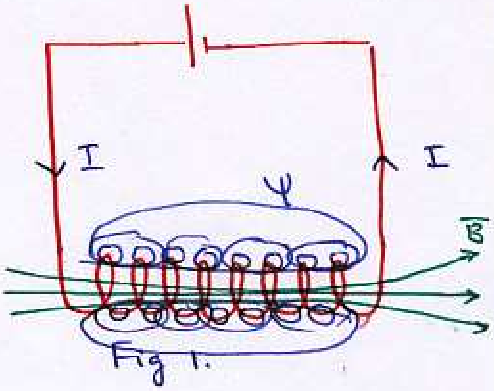


Fig 1.

A circuit (or closed conducting path) carrying current I produces a magnetic field $\bar{B}$ which causes a flux ψ

$$\psi = \int_S \bar{B}.d\bar{s}$$

This flux ψ passes through each turn of the circuit as shown in figure 1. Now let us define flux linkage λ as

$$\lambda = \psi$$

If the coil has N number of turns, then

$$\lambda = N\psi$$

Also, if the medium surrounding the circuit is linear, the flux linkage λ is proportional to the current I producing it; ie.

$$\lambda \alpha I$$

$$\text{(or)} \quad \lambda = LI \quad \Rightarrow \quad L = \frac{\lambda}{I}$$

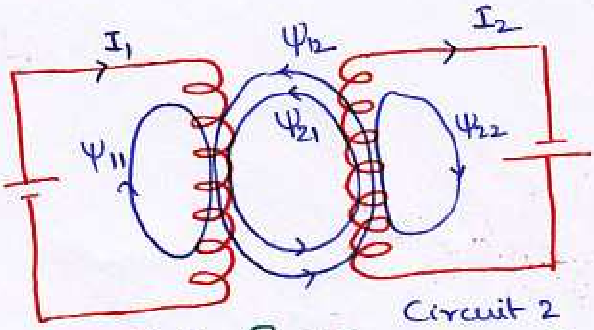
where L is a constant of proportionality called Inductance or Self Inductance of the circuit.

Now we may define Inductance or Self-inductance as the ratio of total flux linkage to the current flowing through the coil.

$$\boxed{L = \frac{\lambda}{I} = \frac{N\psi}{I} = \frac{\text{Total flux linkage}}{\text{Current through the coil}}} \quad \text{H or Wb-T/A}$$

Units for Inductance are Henrys (H) but it is a fairly larger unit, inductances are usually expressed in milli-henrys (mH)

Now consider two circuits carrying currents I_1 and I_2 as shown in figure 2, a magnetic interaction exists between the circuits. Let B_1 is the flux produced due to the current I_1 in the circuit 1. Some of the magnetic flux due to B_1 will link with circuit 2, that is, will pass through the surface S_2 bounded by circuit 2, which produces mutual flux Ψ_{12}



Circuit 1 Figure 2

$$\therefore \Psi_{12} = \int_{S_2} \bar{B}_1 . d\bar{S}_2$$

If circuit 2 contains N_2 number of turns, ~~then~~ the flux linkage

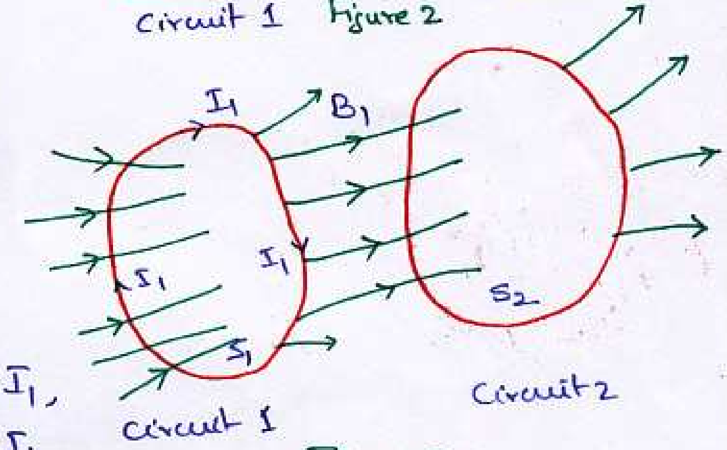


Figure 3

$$\lambda_{12} = N_2\Psi_{12}$$

From Biot_Savart's Law, we see that B_1 is directly proportional to I_1, hence Ψ_{12} is also proportional to I_1. Therefore, Total flux linkage λ_{12} is portional to current I_1 ~~$\Psi_{12} \propto I_1$~~ $\lambda_{12} \propto I_1$

~~$\lambda_{12}\ \Psi_{12} = M_{12} I_1$~~ $\lambda_{12} = M_{12} I_1$

or ~~$M_{12} = \frac{\Psi_{12}}{I_1}$~~ $M_{12} = \frac{\lambda_{12}}{I_1}$

But $\lambda_{12} = \frac{N_2 \Psi_{12}}{\cancel{I_1}}$

$$\therefore \boxed{M_{12} = \frac{N_2 \Psi_{12}}{I_1}\ H}$$

where M_{12} is a proportionality constant called Mutual Inductance between circuit 1 and circuit 2.

Similarly, $\boxed{M_{21} = \frac{N_1 \Psi_{21}}{I_2}\ H}$

If the surrounding medium is linear, then $M_{12} = M_{21}$

Now the mutual inductance (M) between two coils is defined as the ratio of the total flux linkage in one coil to the current in the other coil.

⊛ Self-Inductance

$$\frac{N_1\Psi_{11}}{I_1} = L_{11} = L_1$$

$$\frac{N_2\Psi_{22}}{I_2} = L_{22} = L_2$$

are called self inductance and defined as the ratio of the magnetic flux linkage to the current in the loop itself.

Coefficient of Coupling

The coefficient of coupling between two coils is defined as the ratio of the total flux linkage between two coils to the flux produced by one coil.

$$k = \frac{\text{Total flux linkage between coil 1 and coil 2}}{\text{Flux produced by coil 1 or coil 2}}$$

$$k = \frac{\psi_{12}}{\psi_1} = \frac{\psi_{21}}{\psi_2}$$

It is also defined as $k = \frac{M}{\sqrt{L_1 L_2}}$ where $M = M_{12} = M_{21}$

$L_1 = L_{11}$

$L_2 = L_{22}$

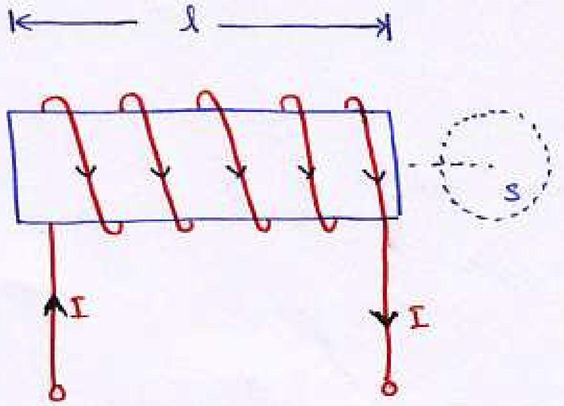
The range of k is $0 \leq k < 1$.

$$k = \begin{cases} 0 & \text{No coupling} \\ 1 & \text{perfect coupling} \end{cases}$$

Comparison between Self and Mutual Inductance

	Self - Inductance		Mutual - Inductance
1.	It is the ratio of total flux linkage to the current flowing through the coil itself. $L = \frac{N\psi}{I}$ H	1.	It is the ratio of the total flux linkage in one coil to the current in other coil. $M_{12} = \frac{N_1\psi_{21}}{I_2}$ (or) $M_{21} = \frac{N_2\psi_{12}}{I_1}$
2.	It is associated with one coil only	2.	It is associated with more than one coil.
3.	It is denoted by L	3.	It is denoted M.
4.	Self inductance of a coil can be increased by increasing the number of turns (N), or cross-section of conductor (A) or by selecting the material of high permeability (μ)	4.	The mutual inductance of a coil can be increased by increasing the relative permeability (μ_r)(or) maintaining low spacing between the two coils.
5.	It depends on the current I passing through its own coil.	5.	It depends on the current in the neighbouring coil.

Inductance of a Solenoid.



Consider a solenoid of N turns, length l, and a cross-section S.
Let the current flowing through solenoid be I amperes. Therefore, flux density produced inside the solenoid is given by

$$B = \mu H = \frac{\mu N I}{l} \quad T$$

Now the total flux linkage is given by

$$\lambda = N\psi = NBS \qquad \left(\because \psi = BS \right.$$

$$\text{or } \psi = \int_s \bar{B}.d\bar{s}$$

$$= |B|\int|ds|\cos\theta \quad \text{\& } \theta = 0 \text{ as } \bar{B} \text{ \& } \bar{s} \text{ are in the same direction}$$

$$\left. = |B||S| \right)$$

$$= N.\left(\frac{\mu N I}{l}\right)S$$

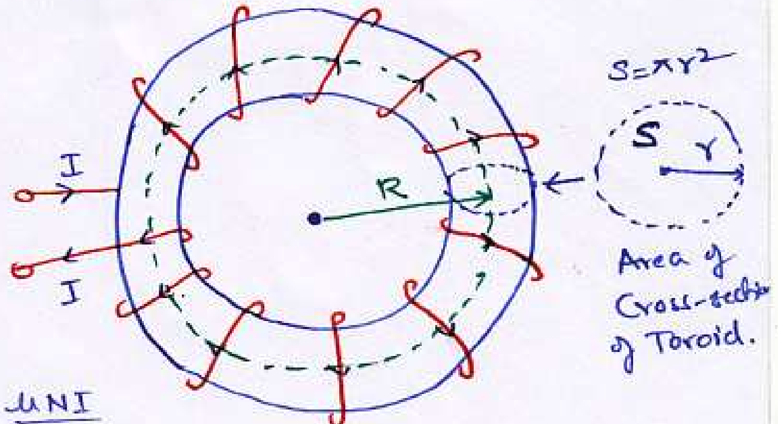
$$= \frac{\mu N^2 I S}{l}$$

Inductance of the solenoid is

$$L = \frac{\text{Total flux linkage}}{\text{Total current}} = \frac{\mu N^2 I S/l}{I}$$

$$\boxed{L = \frac{\mu N^2 S}{l} \quad H}$$

Inductance of Toroid



Consider a toroidal ring with N turns and carrying current I. Let the radius of the toroid be R as shown in figure. Now the magnetic flux produced by the current I inside the toroid is given by

$$B = \mu H = \frac{\mu N I}{2\pi R} \qquad \left(\because B = \frac{\mu N I}{2\pi\rho} \text{ but } \rho = R\right)$$

Total flux linkage of toroidal ring is given by

$$\lambda = N\psi$$

But $\psi = BS$

$$\therefore \lambda = NBS$$

Substituting for B $\lambda = \frac{N \mu N I S}{2\pi R}$

$$= \frac{\mu N^2 I S}{2\pi R}$$

Inductance of the Toroid is

$$L = \frac{\lambda}{I} = \frac{\frac{\mu N^2 \cancel{I} S}{2\pi R}}{\cancel{I}} = \frac{\mu N^2 S}{2\pi R} \, H$$

where $S = \cancel{\pi R}\,\pi r^2$

$$\therefore \boxed{L = \frac{\mu N^2 S}{2\pi R} = \frac{\mu N^2 r^2}{2R} \quad H}$$

Magnetic Scalar & Vector Potential

In electrostatic field problems, electric potential 'V' is related to electric field $\overline{E}$ ($\overline{E} = -\nabla V$). Similarly we can define magnetic potential with magnetic field. Infact, the magnetic potential could be scalar 'V_m' or vector $\overline{A}$.

Scalar Magnetic Potential:

Def: Scalar Magnetic Potential is defined as the line integral of the magnetic field intensity along a given path, where current density is zero.

$$\boxed{V_m = -\int_L \overline{H} \cdot d\overline{L} \quad (\text{or}) \quad V_{mAB} = -\int_A^B \overline{H} \cdot d\overline{L}}$$

We know that
$$\nabla \cdot \overline{B} = 0$$
or
$$\nabla \cdot \mu \overline{H} = 0$$
Since $\mu \neq 0$
$$\nabla \cdot \overline{H} = 0$$

But the negative gradient of the scalar magnetic potential is equal to magnetic field intensity at which the current density $\overline{J} = 0$

that is $\overline{H} = -\nabla V_m$ if $\overline{J} = 0$

$\therefore$ $\nabla \cdot \overline{H} = 0$ can be written as

$$\nabla \cdot (-\nabla V_m) = 0$$

$$-\nabla^2 V_m = 0$$

$$\boxed{\text{or } \nabla^2 V_m = 0 \quad \text{for } J = 0} \rightarrow$$ Laplace's Equation for Scalar Magnetic Field

Note: Scalar Magnetic Potential is used to find the magnetic field intensity and flux in a region where there is no current exists. This happens in case of fields due to permanent magnets.

Vector Magnetic Potential ($\bar{A}$)

Def:

In a given magnetic field, the curl of the vector magnetic potential $\bar{A}$ is equal to the magnetic flux density.

$$\boxed{\bar{B} = \nabla \times \bar{A}}$$

where $\bar{A}$ is the vector magnetic potential wb/m

From Ampere's law

$$\nabla \times \bar{H} = \bar{J}$$

$$\nabla \times \frac{\bar{B}}{\mu} = \bar{J} \Rightarrow \nabla \times \bar{B} = \mu \bar{J} \longrightarrow ①$$

But $\bar{B} = \nabla \times \bar{A}$

Substituting into eq ①

$$\nabla \times \nabla \times \bar{A} = \mu \bar{J}$$

$$\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} = \mu \bar{J}$$

From vector identities $\nabla(\nabla \cdot \bar{A}) = 0$

$$\therefore \cancel{A} - \nabla^2 \bar{A} = \mu \bar{J}$$

or $\boxed{\nabla^2 \bar{A} = -\mu \bar{J}}$ — Poisson's Equation

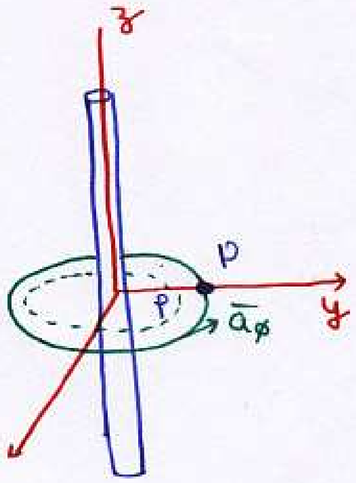
If $\bar{J} = 0$, then

$\boxed{\nabla^2 \bar{A} = 0}$ — Laplace's Equation

Note: The vector Magnetic potential is used to obtain radiation characteristics of antennas.

Vector Magnetic Potential in the Field due to Infinite Line Current.

Consider an infinite long current carrying conductor along the z-axis as shown in figure.



The magnetic field at P is given by

$$\bar{H} = \frac{I}{2\pi\rho}\bar{a}_\phi$$

$$\therefore \bar{B} = \frac{\mu I}{2\pi\rho}\bar{a}_\phi$$

and vector magnetic potential $\bar{A}$ is related to $\bar{B}$ as

$$\bar{B} = \nabla\times\bar{A} = \frac{\mu I}{2\pi\rho}\bar{a}_\phi$$

$\nabla\times\bar{A}$ can be written as

$$\nabla\times\bar{A} = \frac{1}{\rho}\begin{vmatrix} \bar{a}_\rho & \rho\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} = \frac{\mu I}{2\pi\rho}\bar{a}_\phi$$

Expanding and equating $\bar{a}_\phi$ components

$$\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\bar{a}_\phi = \frac{\mu I}{2\pi\rho}\bar{a}_\rho$$

$$\Rightarrow \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho} = \frac{\mu I}{2\pi\rho}$$

Since $\bar{B}$ and $\bar{A}$ are constant along z-direction $\frac{\partial A_\rho}{\partial z} = 0$

$$\therefore \frac{-\partial A_z}{\partial\rho} = \frac{\mu I}{2\pi\rho}$$

$$\partial A_z = \frac{-\mu I}{2\pi\rho}d\rho$$

Integrating on both sides

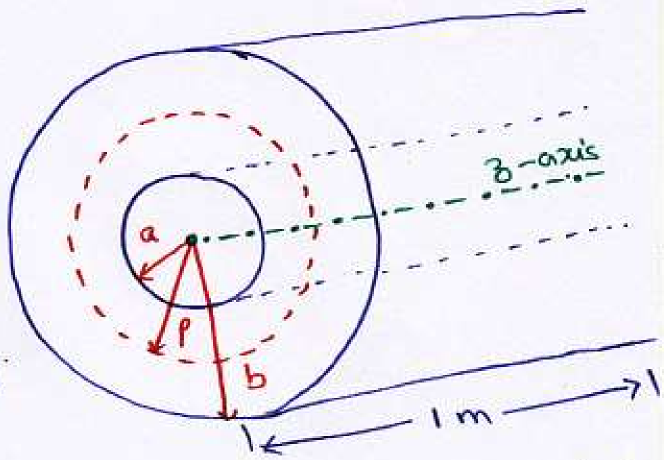
$$A_z = \int dA_z = -\frac{\mu I}{2\pi}\int\frac{d\rho}{\rho}$$

$$\therefore \bar{A} = -\frac{\mu I}{2\pi}\int\frac{d\rho}{\rho}\,\bar{a}$$

or $\boxed{\bar{A} = \frac{\mu I}{2\pi}\int\frac{d\rho}{\rho}(-\bar{a})}$ (or) $A \doteq \frac{\mu I}{2\pi}\int\frac{d\rho}{\rho}$ wb/m

Inductance of A Co-axial Cable:

Consider a co-axial cable has a inner conductor of radius 'a' and a very thin outer conductor of inner radius 'b'. Assume that a current I flows in the inner conductor and returns via the outer conductor in the other direction. Because of the cylindrical symmetry. $\overline{B}$ has only a_ϕ component with different expressions in the two regions. (i) inside the inner conductor and (ii) between the inner and outer conductors.



Let 'ρ' be the radius of the Amperian path as shown in figure.

(i) Inside the conductor $0 \le \rho \le a$,

Applying Ampere's law

$$\oint \overline{H} \cdot d\overline{l} = I_{end}$$

$$I_{end} = I \frac{\rho^2}{a^2} \quad \text{and} \quad \oint H_1 dl_1 = H(2\pi\rho)$$

$$\therefore \quad H_1 = \frac{I\rho}{2\pi a^2}$$

$$\overline{B}_1 = \frac{\mu I \rho}{2\pi a^2} \cdot \overline{a_\phi}$$

(ii) Between the inner & outer conductors

$$a \le \rho \le a$$

Applying Ampere's law $\oint \overline{H_2} \cdot d\overline{l_2} = I_{end}$

$$H_2(2\pi\rho) = I \Rightarrow \overline{H_2} = \frac{I}{2\pi\rho} \overline{a_\phi}$$

$$\overline{B_2} = \frac{\mu I}{2\pi\rho} \overline{a_\phi}$$

Total flux linkage in both the regions

$$\psi = \psi_1 + \psi_2 = \int_{S_1} \overline{B_1} \cdot d\overline{S_1} + \int_{S_2} \overline{B_2} \cdot d\overline{S_2}$$

$$d\overline{S_1} = \frac{\pi \rho^2}{\pi a^2} d\rho \, dz \, \overline{a_\phi} = \frac{\rho^2}{a^2} d\rho dz a_\phi$$

$$dS_2 = d\rho \, dz \, \overline{a_\phi}$$

$$\therefore \psi = \int_{z=0}^{l} \int_{\rho=0}^{a} \frac{\mu I \rho}{2\pi a^2} \overline{a_\phi} \cdot \frac{\rho^2}{a^2} d\rho \, dz \, \overline{a_\phi} + \int_{z=0}^{l} \int_{\rho=a}^{b} \frac{\mu I}{2\pi\rho} \overline{a_\phi} \cdot d\rho \, dz \, \overline{a_\phi}$$

$$\psi = \frac{\mu I l}{8\pi} + \frac{\mu I l}{2\pi} \ln(b/a)$$

Now, the inductance of the cable

$$L = \frac{\psi}{I} = \frac{\frac{\mu \cancel{I} l}{8\pi} + \frac{\mu \cancel{I} l}{2\pi} \ln(b/a)}{\cancel{I}}$$

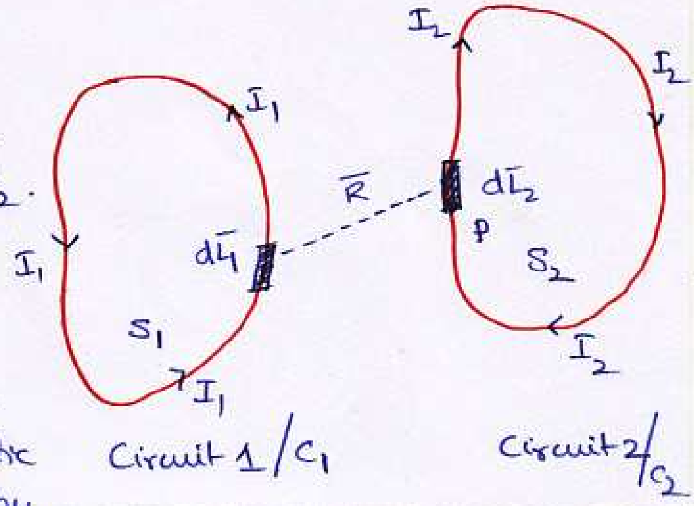
$$L = \frac{\mu l}{2\pi}\left[\frac{1}{4} + \ln(b/a)\right] H$$

If $l = 1$ m, then

$$\boxed{L = \frac{\mu}{2\pi}\left[\frac{1}{4} + \ln b/a\right] H/m.}$$

Neuman's Formula for Mutual Inductance

Consider two closed loops C_1 & C_2 of any random shape as shown in figure. Let I_1 and I_2 be the currents flowing through closed loops C_1 & C_2 respectively. Let 'R' be the separation between C_1 & C_2. Let S_1 and S_2 be the surface areas of ~~too~~ C_1 and C_2 respectively.



Consider a point P located along the surface S_2 of loop 2. The vector magnetic potential at P due to loop 1 is given by

$$A_1 = \oint_{C_1} \frac{\mu I_1 dL_1}{4\pi R} = \frac{\mu I_1}{4\pi}\oint_{C_1}\frac{dL_1}{R}$$

B_1, the magnetic flux density can be expressed interms of the vector magnetic potential A_1 as given below

$$\bar{B}_1 = \nabla\times\bar{A}_1$$

Let ψ_{12} be the flux linking coil C_2 due to current in coil C_1.

$$\therefore \lambda_{12} = N_2\psi_{12} = \int_{S_2}\bar{B}_1\cdot d\bar{S}_2$$

Assuming that both the circuits having single turn ie $N_1 = N_2 = 1$

$$\lambda_{12} = \psi_{12} = \int_{S_2}\bar{B}_1\cdot d\bar{S}_2 = \int_{S_2}(\nabla\times\bar{A}_1)\cdot d\bar{S}_2$$

$$\lambda_{21} = \psi_{21} = \int \bar{B}_2\cdot d\bar{S}_1 = \int_{S_1}(\nabla\times\bar{A}_2)\cdot d\bar{S}_1$$

Converting surface integral into line integral

$$\int_{S_2}(\nabla\times\bar{A}_1)\cdot d\bar{S}_2 = \oint_{C_2}\bar{A}_1\cdot d\bar{L}_2$$

$$\int_{S_1}(\nabla\times\bar{A}_2)\cdot d\bar{S}_1 = \oint_{C_1}\bar{A}_2\cdot d\bar{L}_1$$

$$\therefore \lambda_{12} = \oint_{C_2}\bar{A}_1\cdot d\bar{L}_2$$

$$= \oint_{C_2}\left[\frac{\mu I_1}{4\pi}\oint_{C_1}\frac{d\bar{L}_1}{R}\cdot d\bar{L}_2\right]$$

$$\lambda_{12} = \frac{\mu I_1}{4\pi}\oint_{C_2}\oint_{C_1}\frac{d\bar{L}_1\cdot d\bar{L}_2}{R}$$

and $$\lambda_{21} = \oint_{C_1}\bar{A}_2\cdot d\bar{L}_1$$

$$= \oint_{C_1}\left[\frac{\mu I_2}{4\pi}\oint_{C_2}\frac{d\bar{L}_2}{R}\cdot d\bar{L}_1\right]$$

$$\lambda_{21} = \frac{\mu I_2}{4\pi}\oint_{C_1}\oint_{C_2}\frac{d\bar{L}_2\cdot d\bar{L}_1}{R}$$

Mutual Inductance

$$M_{12} = \frac{\lambda_{12}}{I_1} \quad \& \quad M_{21} = \frac{\lambda_{21}}{I_2}$$

$\therefore$

~~$M_{12} =$~~

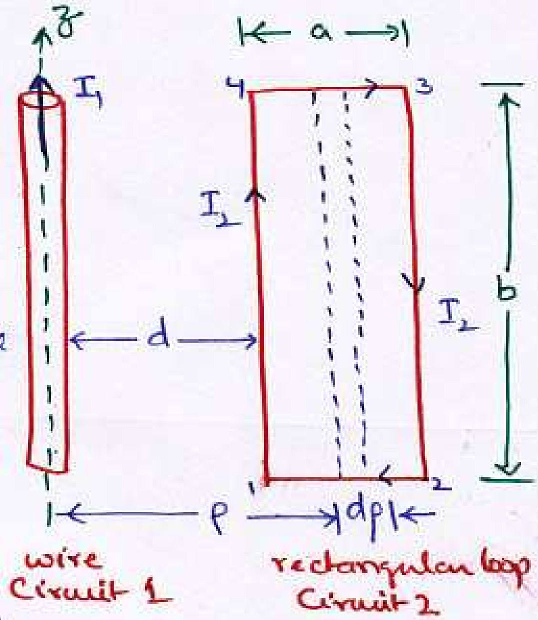
$$\boxed{M_{12} = \frac{\mu}{4\pi}\oint_{C_2}\oint_{C_1}\frac{d\bar{L}_1\cdot d\bar{L}_2}{R} = M_{21}}$$

Neuman's Integrals or

Neuman's Formula.

Mutual Inductance between a Long straight wire and a Rectangular loop lying in the same plane.

Consider a conductor/wire along the z-axis. Let the current flowing in the conductor be I_1 Amps. Consider a rectangular loop 1-2-3-4-1 at a distance 'd' from the the wire and carrying current I_2 as shown in the figure.



Now, magnetic field intensity at a distance ρ from long wire due to current I_1 can be expressed using Ampere's law

$$\oint \bar{H}_1 \cdot d\bar{L}_1 = I_1$$

$$H_{\phi_1} \, 2\pi\rho = I_1 \quad (\because \text{Field will be along } \bar{a}_\phi \text{ only})$$

$$\therefore H_{\phi_1} = \frac{I_1}{2\pi\rho} \longrightarrow ①$$

$$\therefore \bar{B}_1 = B_{\phi_1}\bar{a}_\phi = \mu H_{\phi_1}\bar{a}_\phi$$

$$B_1 = \frac{\mu I_1}{2\pi\rho}\bar{a}_\phi$$

Flux linkage λ_{12} in circuit 2 due to current I_1 in circuit 1 or wire is given by

$$\lambda_{12} = \int_{S_2} \bar{B}_1 \cdot d\bar{S}_2$$

$$= \int_d^{d+a} \left(\frac{\mu I_1}{2\pi\rho}\bar{a}_\phi\right) \cdot (b\,d\rho)\bar{a}_\phi$$

$$= \frac{\mu I_1 b}{2\pi}\int_d^{d+a} \frac{d\rho}{\rho}$$

$$\therefore \lambda_{12} = \frac{\mu I_1 b}{2\pi} \ln\left(1+\frac{a}{d}\right) \quad \text{wb}$$

Assuming single turn circuit

$$M_{12} = \frac{\lambda_{12}}{I_1}$$

$$= \frac{\frac{\mu \cancel{I_1} b}{2\pi}\ln\left(1+\frac{a}{d}\right)}{\cancel{I_1}} \quad H$$

$$\therefore \boxed{M_{12} = \frac{\mu b}{2\pi}\ln\left(1+\frac{a}{d}\right) \quad H}$$

Energy Stored in a Magnetic Field.

Similar to capacitor, an inductor is also an energy storing element. In a capacitor, the energy is stored in the form of electrostatic field, while in an inductor in the form of magnetic field.

Consider a differential volume in a magnetic field $\bar{B}$ as shown in figure. Consider that the top and bottom surfaces of differential volume as conducting sheets with current ΔI.

Now the inductance due to the differential volume and flux density $\bar{B}$ is given by

$$\Delta L = \frac{\Delta \psi}{\Delta I} = \frac{B \Delta S}{\Delta I}$$

$$\left(\because \psi = \int_S \bar{B}\cdot d\bar{S} \Rightarrow \Delta\psi = \bar{B}\cdot\Delta\bar{S} = B\Delta S\right)$$

where $\Delta S = \Delta x \Delta z$ differential area

$$\therefore \Delta L = \frac{B \Delta x \Delta z}{\Delta I} \longrightarrow ①$$

Now the differential current ΔI can be expressed interms of magnetic field $\bar{H}$ as

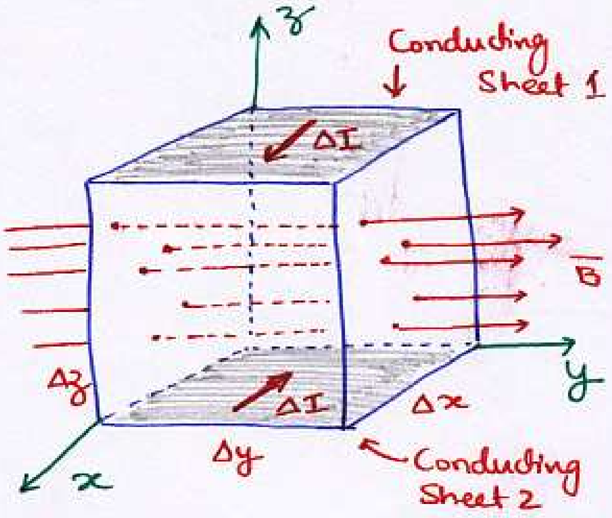
$$\Delta I = \bar{H}\cdot d\bar{L} = \bar{H}\cdot\Delta\bar{L} \longrightarrow ②$$
$$= H\Delta y$$

The energy stored in an inductor of differential volume is given by

$$\Delta W_m = \frac{1}{2}\Delta L \Delta I^2 \quad \left(\because W_m = \frac{1}{2}LI^2\right) \longrightarrow ③$$

Substituting ① & ② into equation ③

$$\Delta W_m = \frac{1}{2}\frac{B\Delta x\Delta z}{\Delta I}(H\Delta y)^2$$
$$= \frac{1}{2}\frac{B\Delta x\Delta z}{\cancel{H\Delta y}} H^{\cancel{2}}\Delta y^{\cancel{2}}$$



$$\Delta W_m = \frac{1}{2} BH \Delta x \Delta y \Delta z$$
$$= \frac{1}{2} BH \Delta v$$
$$(\because \Delta x \Delta y \Delta z = \Delta v \text{ volume})$$

or
$$\frac{\Delta W_m}{\Delta v} = \frac{1}{2}BH$$

As $\Delta v \to 0$

$$w_m = \lim_{\Delta v \to 0}\frac{\Delta W_m}{\Delta v} = \frac{1}{2}BH \ J/m^3$$

$$\Rightarrow \boxed{\begin{aligned} w_m &= \frac{1}{2}BH \ J/m^3 \\ \text{or} \quad w_m &= \frac{1}{2}\mu H^2 \ J/m^3 \\ \text{or} \quad w_m &= \frac{1}{2}\frac{B^2}{\mu} \ J/m^3 \end{aligned}}$$

w_m — is the magnetostatic Energy Density.

In linear medium

$$W_m = \int_v w_m dv$$

or
$$\boxed{W_m = \frac{1}{2}\int \bar{B}\cdot\bar{H}\, dv = \frac{1}{2}\int \mu H^2 dv}$$

Magnetic Materials : Types

Magnetic Materials can be classified into six different types :
① Diamagnetic, ② paramagnetic, ③. ferromagnetic
④. Antiferromagnetic ⑤. ferrimagnetic and ⑥. superparamagnetic.

This classification is based on their material properties, magnetic susceptbility(χ_m ~~of~~ or relative permeability.(μ_r)

A comparision between three types of materials is given below:

	Diamagnetic		Paramagnetic		Ferromagnetic.
1.	Permanent Magnetic moment of each atom is zero	1.	Permanent magnetic moment is not zero	1.	very large permanent magnetic moment.
2.	Flux density $\bar{B} = 0$	2.	$\bar{B} = \mu\bar{H}$	2.	$\bar{B} \neq \mu\bar{H}$, since μ depends on $\bar{B}$
3.	Linear and non magnetic characteristics	3.	Linear and non magnetic characteristics	3.	Non-linear and magnetic characteristics
4.	Weak magnetic properties	4.	Slightly stronger magnetic properties	4.	Very strong magnetic properties
5.	Diamagnetism is independent of temperature $\mu_r \leq 1$ $\chi_m < 0$ (very small $\chi_m \simeq -10^{-5}$)	5.	Paramagnetism depends on temparature $\mu_r \geq 1$ $\chi_m > 0$ $(10^{-5} \leq \chi_m \leq 10^{-3})$	5.	ferromagnetism depends on temperature. If temperature rises above curie temperature, magnets lose their magnetic property $\mu_r >> 1$ $\chi_m >> 0$ (very large χ_m)
6.	Examples: Copper, lead, silicon, Diamond, bismuth etc.	6.	Examples: Air, tungsten, platinum, potassium etc	6.	Examples: Iron, Nickel, cobalt etc.

Magnetic Circuits:

The medium through which the magnetic flux is flowing in a closed loop is called a magnetic circuit. When magnetic flux is flowing through a medium, reluctance is the property of the medium which opposes the flow of the magnetic flux.

Def: Reluctance is defined as the ratio of magnetomotive force to the total flux

$$\mathbb{R} = \frac{mmf}{\psi} = \frac{NI}{\psi} \quad AT/wb$$

or

$$\boxed{\mathbb{R} = \frac{NI}{\psi} = \frac{l}{\mu S}}$$

N – Number of turns
S – Area of Cross-section
l – length of the coil.

Reluctance is analogus to resistance in an electric circuit. Since reluctance is inversely proportional to permeability, its value is very high for air and non-magnetic materials. For ferromagnetic materials such as steel or iron, the reluctance is small.

Relationship between Reluctance and Inductance

When a current I is passing through a coil of N turns, the flux produced is ψ wb. Then the self inductance of the magnetic path is

$$L = \frac{N\psi}{I} \longrightarrow ①$$

and the reluctance of the magnetic path is

$$\mathbb{R} = \frac{NI}{\psi}$$

$$\text{or } \psi = \frac{NI}{\mathbb{R}}$$ Substituting in to eq ①

$$\boxed{\therefore L = \frac{N}{I}\,\frac{NI}{\mathbb{R}} = \frac{N^2}{\mathbb{R}} \; H}$$

If there are two coils of turns N_1 and N_2, then the mutual inductance between the two coils is

$$\boxed{M = \frac{N_1 N_2}{\mathbb{R}}}$$

Permanent Magnets : Characteristics & Applications.

Applications

Permanent magnets are a vital part of modern life. They are found in or used to produce almost every modern convenience today, from Speakers in mobile phones to electric motors in hybrid cars; air conditioners and washing machines. Permanent magnets are used increasingly in technological applications, including travelling wave wave tubes, Hall Effect sensors, high temperature-resistant permanent magnets, thin-film coating equipment and flywheel storage systems.

In all these applications, it is important for the designed permanent magnet to be of high strength, resistive to corrosion, and resistive to demagnetization due to excessive heat.

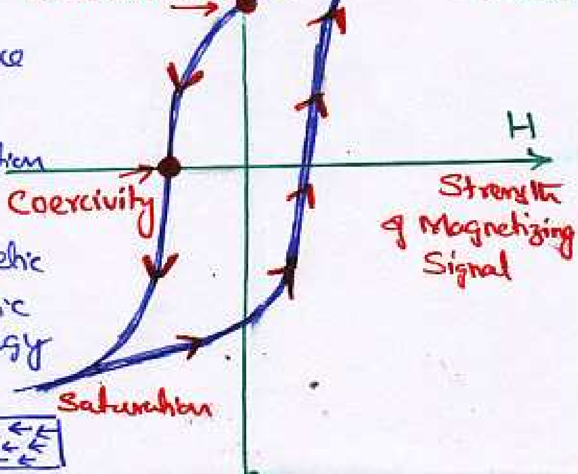
There are a number of major families of permanent magnets available for designers, ranging from ferrite, known for its low cost low energy, to rare earth materials, which are more expensive and offer higher performance.

Characteristics:

A good permanent magnet should produce a high magnetic field with a low mass, and should be stable against the influences which would demagnetize it. The desirable properties of such magnets are typically stated interms of the

(i) remanence: a mesure of the ~~remo~~ remaining ~~in~~ magnetisation when the driving field is dropped to zero and

(ii) Coercivity: a measure of the reverse field needed to drive the magnetization to zero after being saturated.



If an alternating magnetic field is applied to a material, its magnetization will ~~trace~~ trace out a loop called haysteresis loop. (Figure). The lack of retraccability of the magnetization curve is the property called hysteresis and is related to the existance of magnetic domains in the material. Once the magnetic domains are oriented, it takes some energy to turn them again. Some compositions of ferromagnetic materials will retain an imposed magnetism indefinitely are useful as permanent magnets.

UNIT-8: Time-Varying Fields

Faraday's law : The electromotive force (emf) induced in a closed path (or circuit) is proportional to rate of change of magnetic flux enclosed by the closed path (or linked with the circuit).

① He observed that when a closed path moves in a magnetic field, current is generated and hence emf. The same observation he made with closed loop kept fixed and the magnetic field was varied. The effect is commonly called electromagnetic Induction.

Faraday's law can be stated as follows:

$$e = +N\frac{d\phi}{dt} \text{ volts}$$

N – no of turns in the ckt
e – Induced emf

Lenz's law: The direction of induced emf is such that it opposes the cause producing it, i.e. changes the magnetic flux. (induced emf acts to produce an opposite flux)

$$e = -N\frac{d\phi}{dt} \text{ volts}$$

Induced emf is a scalar quantity and is measured in volts.

$$\therefore e = \oint \bar{E}.d\bar{l}$$

$$\text{and } \psi = \oint \bar{B}.d\bar{s}$$

(for N=1)

$$\therefore e = -\frac{d\psi}{dt} = -\frac{d}{dt}\int \bar{B}.d\bar{s} = \oint \bar{E}.d\bar{l}$$

$$e = \oint \bar{E}.d\bar{l} = -\frac{d}{dt}\int \bar{B}.d\bar{s}$$

$$= -\int \frac{d\bar{B}}{\partial t}.d\bar{s}$$

$$\Rightarrow \int_s (\nabla \times \bar{E}).ds = -\int_s \frac{\partial \bar{B}}{\partial t}.d\bar{s}$$

$$(\nabla \times \bar{E}).d\bar{s} = -\frac{\partial \bar{B}}{\partial t}.d\bar{s}$$

$$\boxed{\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}}$$ Time varying Fields

$\nabla \times E = 0$ (Electro

$e = \int \bar{E}d\bar{l} = -N\frac{d\phi}{dt}$

$\psi = \phi_m \cos\omega t$

as the flux is alternating

$$\therefore e = -N\frac{d\psi}{dt}$$

$$= -N\frac{d}{dt}(\phi_m \cos\omega t)$$

$$= +N\phi_m \sin\omega t.\omega$$

$$= N\phi_m \omega \sin\omega t \text{ volts.}$$

The value of $\frac{\partial \phi}{\partial t}$ may result from any of the following situations.

1. Time changing flux linking a stationary closed path.
   (or) By having a stationary closed path in a time varying $\overline{B}$ field
   (statically induced emf or transformer emf)
2. Relative motion between a steady flux and a closed path
   (or) By having a time varying closed path in a static $\overline{B}$ field
   dynamically induced emf or motional emf or Generator emf
3. A combination of two (or) by having a time varying closed path in a time varying field $\overline{B}$.
   Both Statically Induced emf & dynamically induced emf

② Dynamically Induced emf

$F = Q\bar{u} \times \bar{B}$ Consider a moving conductor having a total charge Q

$\frac{F}{Q} = \bar{u} \times \bar{B}$ is the Force per unit charge. is called the motional electric Field Intensity

②

$\therefore \bar{E}_m = \bar{u} \times \bar{B}$

Then the motional emf produced by the moving conductor

$$\text{emf} = \oint \bar{E}_m \cdot d\bar{l} = \oint (\bar{u} \times \bar{B}) \cdot d\bar{l} \longrightarrow \text{motional emf along the } x\text{-direction}$$

Conductor of length l. ⟶ $= \int_{l}^{0} uB\,dx$

$= -Bud$ for B is not the function time.

(or) $-Bul$ volts.

If $\bar{B}$ is also changing with time, then we must include both contributions the transformer emf & motional emf

$$\text{ie} \quad \text{emf} = \oint \bar{E} \cdot d\bar{l} = \underbrace{-\int_{s} \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}}_{\text{Transformer emf}} + \underbrace{\oint (\bar{u} \times \bar{B}) \cdot d\bar{l}}_{\text{motional emf.}}$$

Maxwell's Eq in Point Form — Integral Form

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \ (-j\omega \bar{B}) \qquad \oint \bar{E}\cdot d\bar{l} = -\int_s \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s}$$

$$\nabla \times \bar{H} = J + \frac{\partial \bar{D}}{\partial t} \left[\begin{matrix}\sigma \bar{E} + j\omega \bar{D} \\ \text{or } (\bar{J} + j\omega \bar{D})\end{matrix}\right] \qquad \oint \bar{H}\cdot d\bar{l} = I + \int \frac{\partial \bar{D}}{\partial t}\cdot d\bar{s} \ (\sigma + j\omega\epsilon)\int E\cdot$$

$$\nabla \cdot \bar{D} = \rho_v \qquad \oint \bar{D}\cdot d\bar{s} = \int_{vol} \rho_v dv$$

$$\nabla \cdot \bar{B} = 0. \qquad \oint \bar{B}\cdot d\bar{s} = 0$$

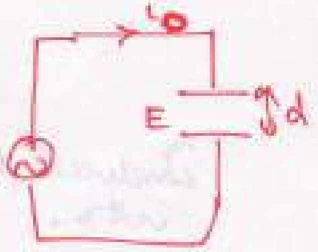
5 ✓

~~Find Ampere the law~~

~~4th Eq with~~

3 ✓

Displacement Current



$$i_c = \frac{dQ}{dt} = c\frac{dV}{dt}$$

and $c = \frac{\epsilon A}{d}$

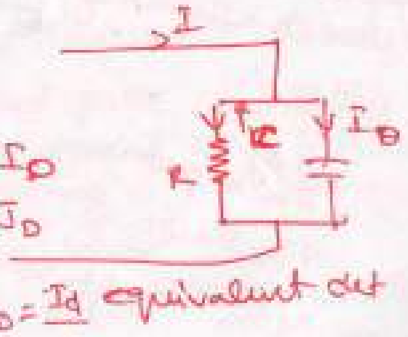
$$\therefore i_c = \frac{\epsilon A}{d}\frac{dV}{dt}.$$

$$= \epsilon A \frac{dE}{dt}, \ (\because E = \frac{V}{d})$$

Displacement current density $\rightarrow J_d = \frac{i_c}{A} = \epsilon\frac{d\bar{E}}{dt} = \frac{d\bar{D}}{dt}$

$$\Rightarrow J_d = \frac{\partial \bar{D}}{\partial t}$$

Displacement current density is not equal to conduction current density



$$I = I_C + I_D$$

$$J = J_C + J_D$$

$J_C = \frac{I_C}{A}$ $J_D = \frac{I_d}{A}$ equivalent ckt

Modified Ampere's Circuital law for Time-Varying Fields.

④ Ampere's law → $\nabla \times \overline{H} = \overline{J}$ $\int (\nabla \times \overline{H}) . d\overline{s}$ $I_{enc} = \int \overline{J} . d\overline{s}$

$\nabla \times \overline{H} = \overline{J}$

taking divergence

$\nabla . (\nabla \times \overline{H}) = \nabla . \overline{J} = 0$ (∵ divergence of curl of any vector field is zero)

But $\nabla . \overline{J} = \frac{-\partial \rho_v}{\partial t}$ Continuity Eq for time varying fields. It does not ($\nabla . J$) vanish to zero in time-varying fields.

∴ The above two equations are in-compatible.

∴ Amperes ckt law is not consistent with time-varying Fields.

∴ Maxwells developed a modification to Ampere's law by substituting Gauss' law in the continuity eq

$$\nabla . \overline{J} = -\frac{\partial \rho_v}{\partial t} \longrightarrow \text{Continuity eq}$$

from Gauss' law $\nabla . \overline{D} = \rho_v$ — Gauss law

$$\nabla . \overline{J} = \frac{-\partial}{\partial t} (\nabla . \overline{D})$$

$$\nabla . \overline{J} = -\nabla . \frac{\partial \overline{D}}{\partial t}$$

or $\nabla . \overline{J} + \nabla . \frac{\partial \overline{D}}{\partial t} = 0$

or $\nabla . \left(J + \frac{\partial \overline{D}}{\partial t} \right) = 0$

or $\int \left(\frac{\partial \overline{D}}{\partial t} + \overline{J} \right) . ds = 0$

Thus The total current through the closed surface is zero

∴ Total current density for time-varying fields is

$$\overline{J_t} = \overline{J} + \frac{\partial \overline{D}}{\partial t}$$

∴ According to Amerer's law

$$\nabla \times \overline{H} = J_t =$$

$$∴ \nabla \times \overline{H} = J + \frac{\partial \overline{D}}{\partial t} = J_c + J_D$$

J_c — Conduction current density

J_D — Displacement current density

$$\frac{|J_c|}{|J_D|} = \frac{|\sigma E|}{|j\omega \epsilon E|} = \frac{\sigma}{\omega \epsilon}$$

$$\int \overline{H} . d\overline{l} = \int \left(J + \frac{\partial \overline{D}}{\partial t} \right) ds = I_{enc}.$$

Maxwell's Eq in Point Form

Integral Form

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad (-j\omega \bar{B}) \qquad \oint \bar{E} \cdot d\bar{l} = -\int_s \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$$

$$\nabla \times \bar{H} = J + \frac{\partial \bar{D}}{\partial t} \left[\begin{matrix} \sigma \bar{E} + j\omega \bar{D} \\ (J + j\omega \bar{D}) \end{matrix}\right] \qquad \oint \bar{H} \cdot d\bar{l} = I + \int \frac{\partial \bar{D}}{\partial t} \cdot d\bar{s} \quad (\sigma + j\omega\epsilon)\int E \cdot d$$

$$\nabla \cdot \bar{D} = \rho_v \qquad \oint \bar{D} \cdot d\bar{s} = \int_{vol} \rho_v dv$$

$$\nabla \cdot \bar{B} = 0. \qquad \oint \bar{B} \cdot d\bar{s} = 0$$

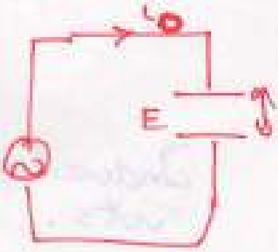
5 ✓

Mod Ampere's ckt law

4th Eq 5th unit

3 ✓

Displacement Current



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Displacement current density $\to J_d = \frac{i_c}{A} = \epsilon \frac{d\bar{E}}{dt} = \frac{d\bar{D}}{dt}$

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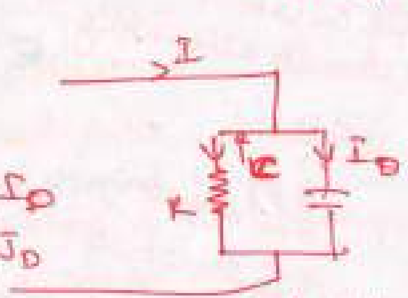

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Poynting Theorem

→ Time varying Fields or dynamic electromagnetic fields constitute the electromagnetic waves.

→ These waves travel through free space or dielectric.

— Eg — radio waves.

— When they travel in free space, the energy transfers from source to destination.

— In case of lumped electrical circuits this energy can be expressed in terms of voltage & current

— In case of electromagnetic waves, the power and energy relationships can be explained in terms of the amplitude of the electric & magnetic fields.

* The resulting theorem is the most fundamental relationship of the EM Theory which is known as Poynting Theorem.

In order to find the power flow associated with an electromagnetic wave, it is necessary to develop a power theorem for the electromagnetic field known as poynting Theorem.

Consider Maxwell's curl eq

$$\nabla \times \overline{H} = \overline{J} \times \frac{\partial \overline{D}}{\partial t}$$

Take scalar product on both sides with $\overline{E}$

$$\overline{E} \cdot \nabla \times \overline{H} = \overline{E} \cdot \overline{J} + \overline{E} \cdot \frac{\partial \overline{D}}{\partial t}$$

$$\nabla \cdot (\overline{E} \times \overline{H}) = -E \cdot \nabla \times \overline{H} + H \cdot \nabla \times E$$

(vector identity)

$$\therefore H \cdot \nabla \times \overline{E} - \nabla \cdot (\overline{E} \times \overline{H}) = \overline{J} \cdot \overline{E} + E \cdot \frac{\partial \overline{D}}{\partial t}$$

But $\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$

$$\therefore -H \cdot \frac{\partial \overline{B}}{\partial t} - \nabla \cdot (\overline{E} \times \overline{H}) = \overline{J} \cdot E + \overline{E} \cdot \frac{\partial \overline{D}}{\partial t}$$

$$-\nabla \cdot (\overline{E} \times \overline{H}) = \overline{J} \cdot E + \epsilon \overline{E} \cdot \frac{\partial \overline{E}}{\partial t} + \mu \overline{H} \cdot \frac{\partial \overline{H}}{\partial t}$$

The cross product $\overline{E} \times \overline{H} = \overline{P}$ or $\overline{S}$ is called poynting vector

from the above eq

$$\epsilon \overline{E} \cdot \frac{\partial \overline{E}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2} \overline{D} \cdot \overline{E}\right)$$

$$\mu \overline{H} \cdot \frac{\partial \overline{H}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2} \overline{B} \cdot \overline{H}\right)$$

$$\therefore -\nabla \cdot (\overline{E} \times \overline{H}) = \overline{J} \cdot \overline{E} + \frac{\partial}{\partial t}\left(\frac{1}{2} \overline{D} \cdot \overline{E}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2} \overline{B} \cdot \overline{H}\right)$$

Integrating throughout with volume

$$-\int_v \nabla \cdot (\overline{E} \times \overline{H}) dv = \int_v \overline{J} \cdot \overline{E}\, dv + \int_v \frac{\partial}{\partial t}\left(\frac{1}{2} \overline{D}\right.$$

$$+ \int_v \frac{\partial}{\partial t}\left(\frac{1}{2} \overline{B} \cdot \overline{H}\right) d$$

applying divergence theorem on LHS

$$\oint_s (\overline{E} \times \overline{H}) \cdot ds = \_\ \_\ \_\ \_$$

eq is known as Poynting