UNIT-I

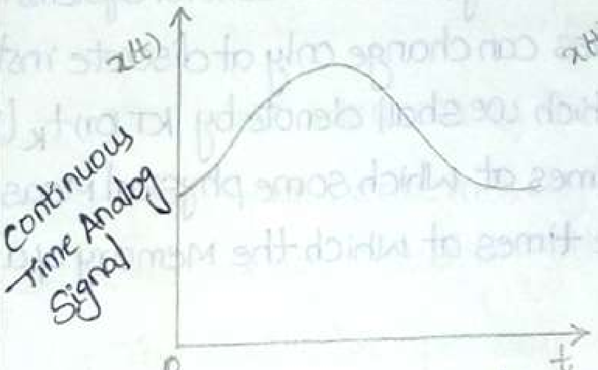
* INTRODUCTION AND SIGNAL PROCESSING *

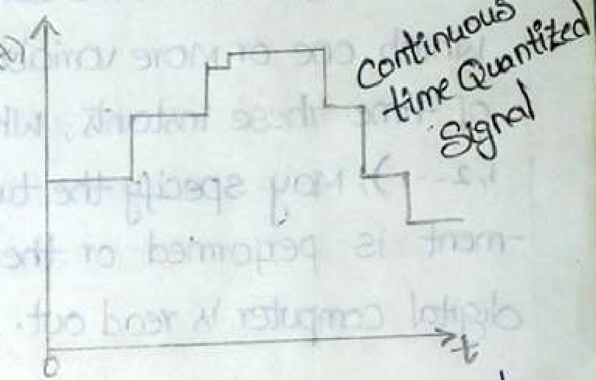
* Introduction to analog & digital control systems :

In recent years significant progress has been Made in the Analysis and design of discrete-data and digital control systems. These systems have gained popularity and importance in industry due in part to the advances Made in digital computers for controls & More recently in Microprocessors & Digital signal processors (DSP).

Discrete-Data & Digital Data control systems differ from the conventional continuous data or analog systems.

Analog signal : An Analog signal is a signal defined over a continuous range of time whose amplitude can Assume a continuous range of values.

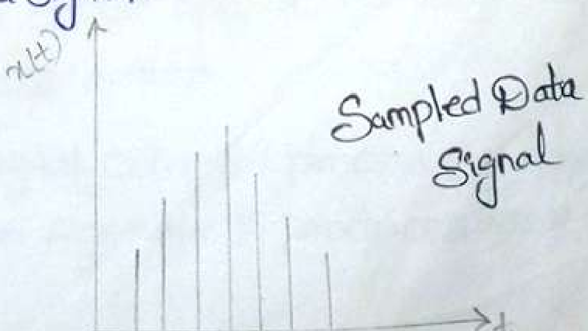




Analog signal is a special case of continuous-time signal.

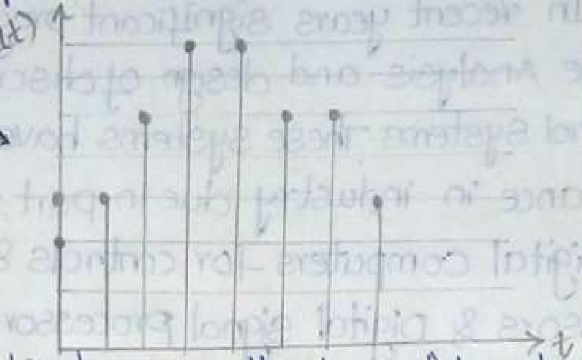
Discrete time signal : A discrete-time signal is a signal defined only at discrete instants of time.

In a discrete-time signal, if the Amplitude can assume a continuous range of values, then the signal is called a sampled-data signal.



A digital signal is a discrete-time signal with quantized amplitude. Such a signal can be represested by a sequence of Numbers. For example in the form of binary Numbers. clearly it is a signal quantized both in amplitude and in time.

Digital signal



In practical usage, the terms "discrete-time" and "digital" are often interchanged. The term "discrete-time" is frequently use in theoritical study, while the term "digital" is used in connection with hardware or software realizations.

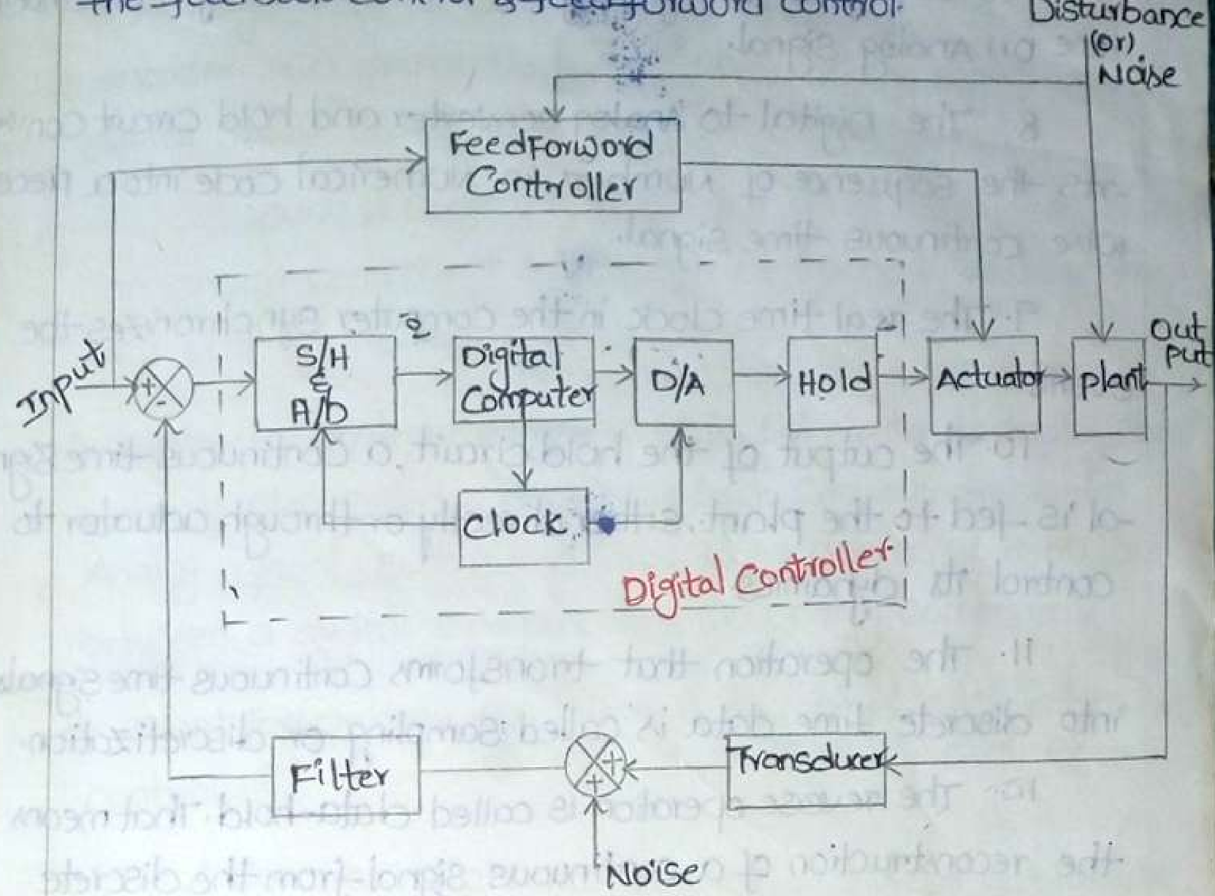
Discrete time control systems & continuous-Time Control System

Discrete-time control systems are control systems in which, one or More variables can change only at discrete instants of time. these instants, which we shall denote by KT (or) t_K ($k=0, 1, 2 \ldots$), May specify the times at which some physical Measure-ment is performed or the times at which the Memory of a digital computer is read out.

Continuous-time systems, whose signals are continuous in time, may be described by differential equations.

3 * BLOCK DIAGRAM OF DIGITAL CONTROL SYSTEMS:

The block diagram of digital control system includes the feedback control & feed forword control.



1. The basic elements of the system are sample and hold, Analog to Digital converter, digital computer, digital to analog converter, Hold, actuator, filters and Transducers etc.

2. The controller operation is controlled by the clock.

3. In such DCS, some points of the system pass signals of varying amplitude in either continuous time or discrete time while other points pass signals in Numerical code.

4. The o/p of the plant is a continuous-time signal.

5. The error signal is converted into digital form by the sample & hold circuit and A/D converter. The conversion is done at sampling rate or time.

6. The digital computer process the sequence of Numbers by means of an Algorithm & produces New sequence of Numbers.

7. At every sampling instant a coded number, usually a binary number consisting of eight or more binary digits must be converted to a physical control signal. Which is usually a continuous time or Analog signal.

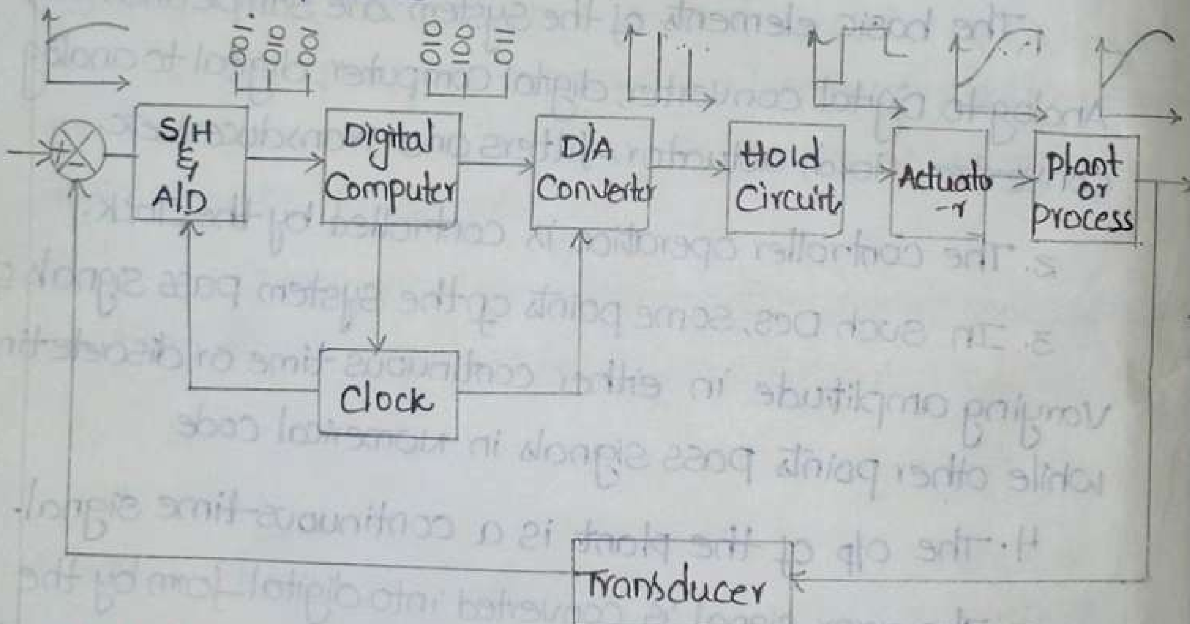
8. The Digital to Analog converter and hold circuit converts the sequence of numbers in numerical code into a piece wise continuous time signal.

9. The real time clock in the computer synchronizes the events.

10. The output of the hold circuit, a continuous-time signal is fed to the plant, either directly or through actuator to control its dynamics.

11. The operation that transforms continuous time signals into discrete time data is called sampling or discretization.

12. The reverse operation is called data-hold. That means the reconstruction of a continuous signal from the discrete time data is possible.



Block Diagram of a digital Control system showing signals in Binary or Graphic form.

Functional blocks in detail

1) Sample & Hold (S|H) : It is the general term used for a sample & hold amplifier. It describes a circuit that receives an analog input signal & holds this signal at a constant value.

5 for a specified period of time. Usually the signal is electrical, but other forms are also possible such as optical & Mechanical.

2. Analog to Digital Converter (A/D): It is also called an encoder is a device that converts an analog signal into a digital signal usually a Numerically coded signal.

such a converter is needed as an Interface between an analog component & a digital component. It is an integral part of a sample & hold circuit.

3 Digital to Analog Converters (D/A): A digital to analog Converter is also called a decoder, which converts digital to Analog signal. such a converter is needed as an interface between a digital component & an analog component.

4. plant [or] process: A plant is any physical object to be controlled.

Ex: A furnace, a chemical reactor, and a set of Machine parts functioning together to perform a particular operation such as a servo system or a space craft.

A process is generally defined as a progressive operation of gradual changes that succeed one another in a relatively fixed way.

5. Transducer: A Transducer is a device that converts an i/p signal into an o/p signal of another form. such as a device that converts a pressure signal into a voltage in output. The o/p signal, in general depends on the past history of the input.

A digital Transducer is one in which the i/p & o/p signals occurs

* Advantages of Digital Control System :

1. Flexibility : An important advantage offered by the DCS is, the flexibility of its modifying controller characteristics, the ability to redesign the controller by changing software is an important feature.

2. Wide selection of Control Algorithms : the digital computers are inexpensive with a limitless computing power,

3. Integral control : It is comprised of many elements like, digital computer, converters of data. The examples of real time applications of information processing & decision making are production planning, scheduling, optimization, operational control etc.

4. Future Generation control systems.

5. Digital components are less susceptible to aging & environmental variations.

6. They are less sensitive to Noise & disturbance

7. Digital processors are More compact & Light weight

8. Single chip Mp's & digital signal processors can be made very versatile & powerful for control ~~co~~ Applications and are cheap.

9. They are more reliable

10. High sensitivity

11. They can handle Non Linear control equations involving complicated computations.

12. Digital Components are rugged in construction.

* Disadvantages :

1. Limitations on Computing speed & signal resolution due to finite wordlength of digital processor.

2. Limitations on computing speed causes time delay in the control loop which may cause instability.

3. The finite word length of digital processor often translates into system instability.

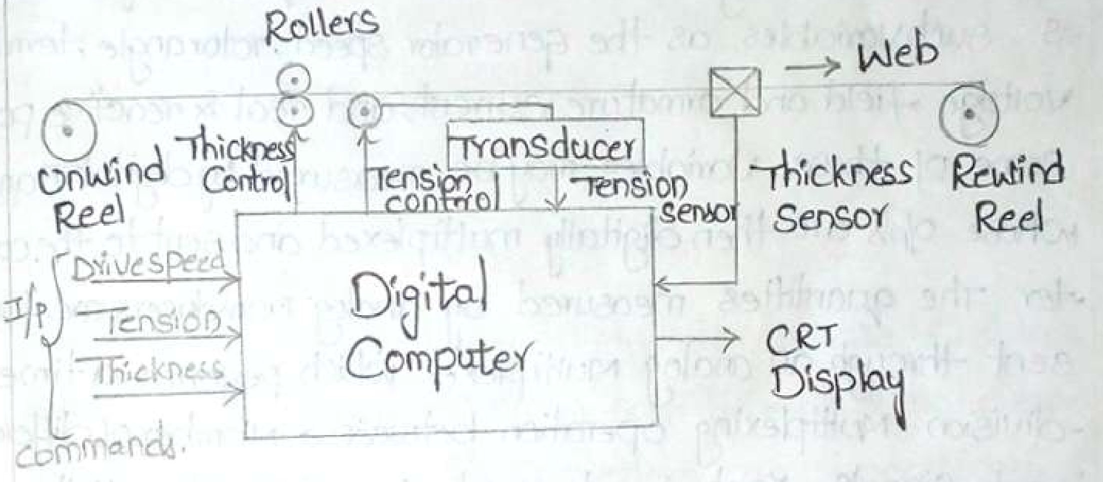
7 * Typical Examples of Discrete-Data & Digital Control System

Different Examples of Digital Control Systems are

1. A Digital Computer-Controlled Rolling Mill Regulating System.
2. A Digital controller for a Turbine and Generator
3. A step motor Control system
4. Microprocessor- Controlled Systems etc.

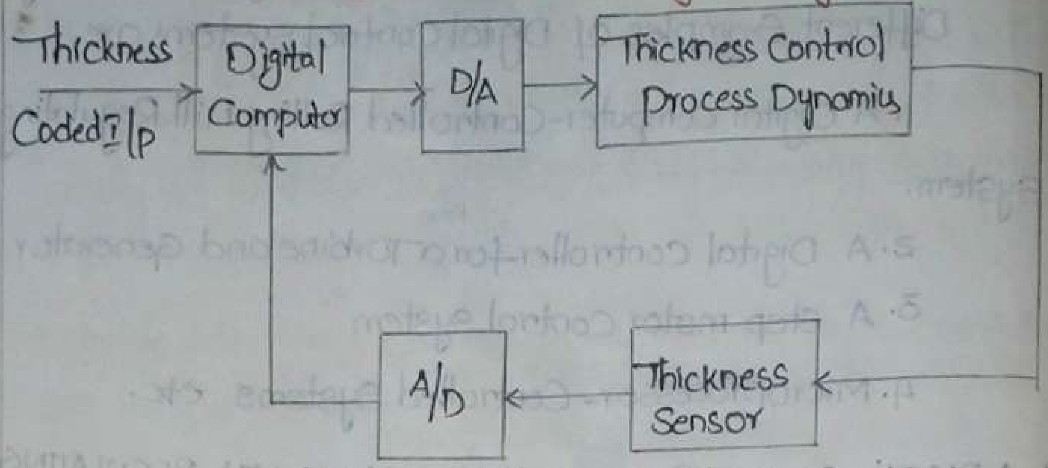
1. A DIGITAL COMPUTER-CONTROLLED ROLLING MILL REGULATING SYSTEM :

Many Industrial process are controlled & Monitored by digital computers and digital transducers. practically all Modern steel rolling mills are regulated and controlled by digital computers.



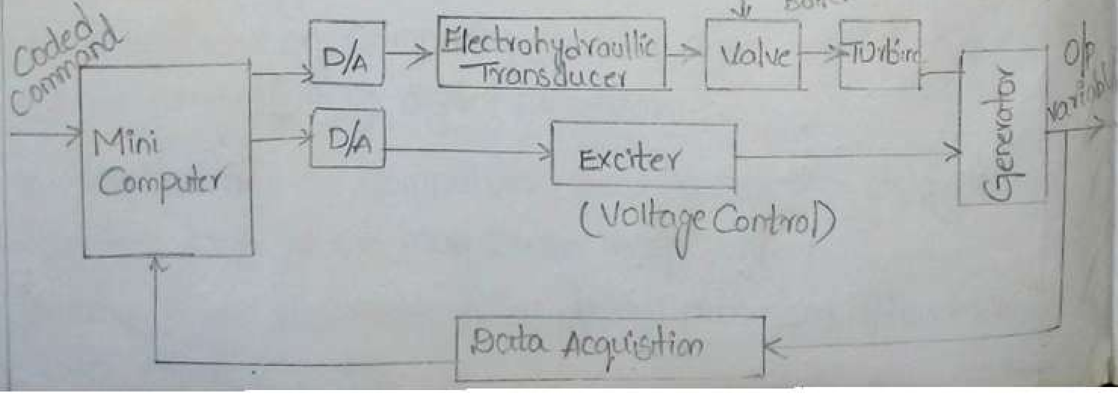
A digital computer-Controlled rolling mill regulating system

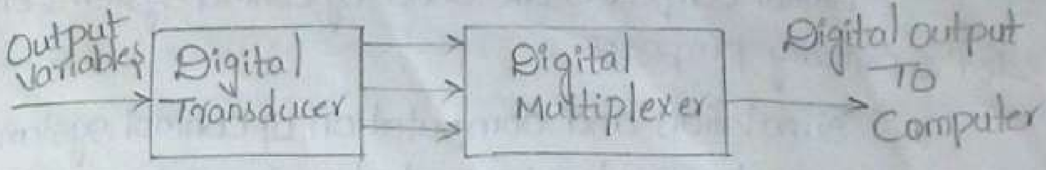
Thickness control in a rolling mill regulating system



2. A Digital Controller for a Turbine & Generator

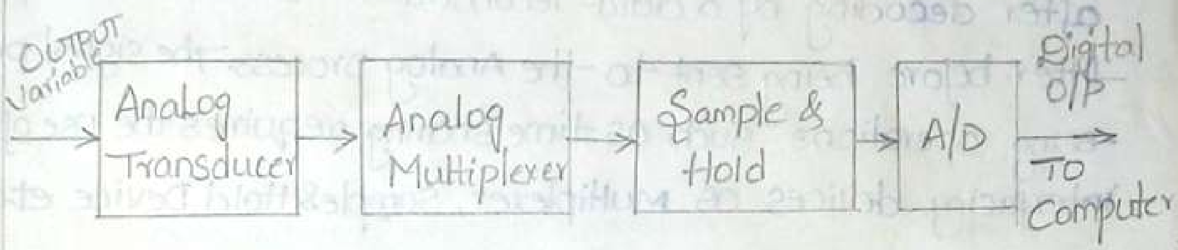
The block Diagram a shows the essential elements of a mini computer system used for speed and voltage control as well as data acquistion of a turbine-Generator unit. The D/A converter forms the interface between the digital computer and speed and voltage controls. The Data-acquistion system Measur-s such variables as the Generator speed, rotor angle, terminal voltage, field and armature current, and real & reactive power Some of these variables may be measured by digital Transducer whose o/p's are then digitally multiplexed and sent to the compu-ter. The quantities measured by analog Transducers are first sent through an analog Multiplexer which performs a time -division Multiplexing operation between a Number of different input signals. Each input-signal channel is sequentially connected to the o/p of the Multiplexer for a specific period of time. The system that follows the Multiplexer is thus time shared between Number of signal channel.





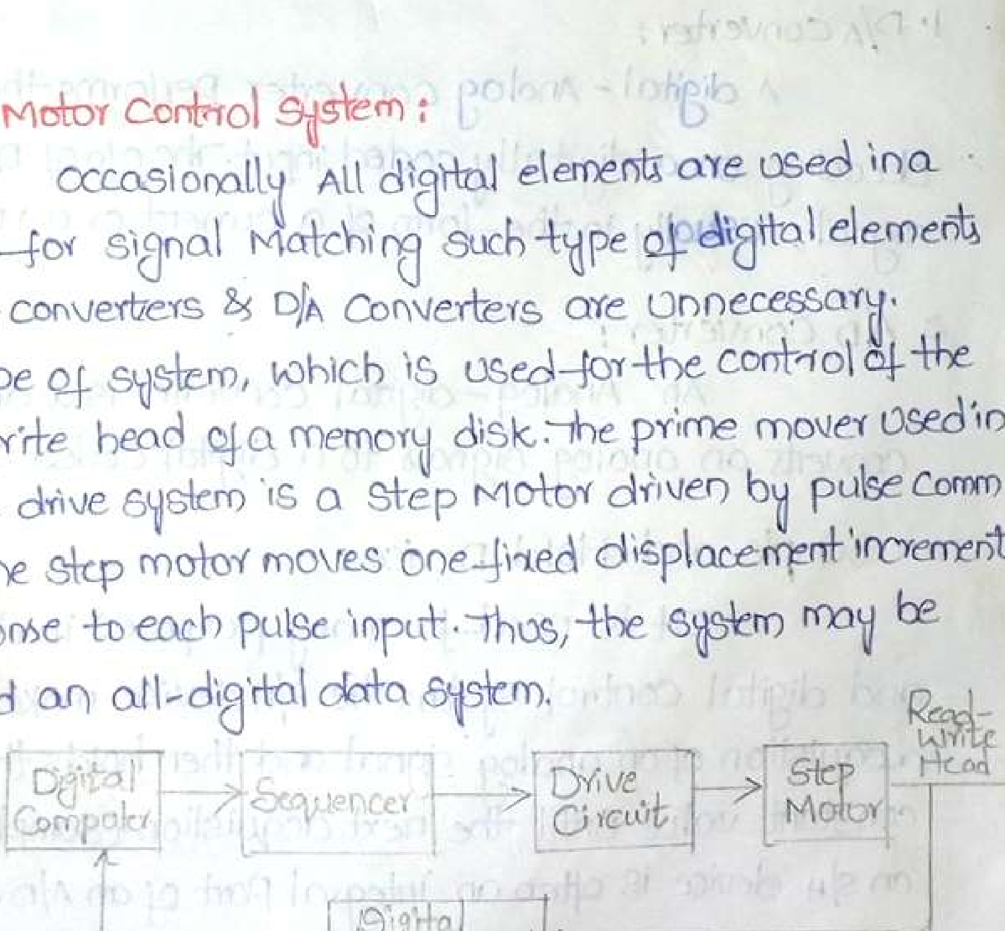
By using Analog Transducer :

The Data acquistion system when the system variables are measured by analog Transducer. The o/p is connected to sample & hold circuit. which samples the o/p of the Multiplexer at a fixed time interval and then holds the signal level at its output until the A/D converter complete its task.



3. A step Motor Control System :

Occasionally All digital elements are used in a system. for signal Matching such type of digital elements like A/D converters & D/A Converters are Unnecessary. such type of system, which is used for the control of the read-write head of a memory disk. The prime mover used in the disk drive system is a step Motor driven by pulse Comm-ands. The step motor moves one fixed displacement increment in response to each pulse input. Thus, the system may be considered an all-digital data system.



*SIGNALS AND PROCESSING:

Digital computers are used by control systems engineers for 2 primary purposes.

1. Simulation and computation of control systems dynamics
2. Digital computers in control systems is as controllers or processors.

Most of the processes to be controlled in the real world contains analog elements. Therefore most of the so called digital control systems usually analog as well as digital signals and the process of signal conversion is essential. So A/D converters, D/A converters are used. since the signal from a digital processor occurs not only as code, but also in the form of a number sequence, the signal must also be smoothed out after decoding by a data-reconstruction device or low pass filter. before being sent to the Analog process. The signal processing operations such as time sharing requires the use of such interfacing devices as Multiplexer, Sample & Hold Device etc.

1. D/A converter:

A digital-Analog converter performs the task of decoding on a digitally coded input. The o/p of D/A is an analog signal, usually in the form of a current or a voltage.

2. A/D converter:

An Analog-digital converter is a device which converts an analog signals to a digital coded signal.

3. Sample-and-Hold Device:

It is used for many purposes in discrete data and digital control system. The S/H device makes the fast acquistion of an analog signal and then holds this signal at a constant value until the next acquistion (sample) is made. an S/H device is often an integral part of an A/D.

4. Multiplexer :

It is used to couple signals from several sources that they can be processed by the same processor or communication channel. The purpose is to time share the processor by all the incoming channels.

Digital signals :

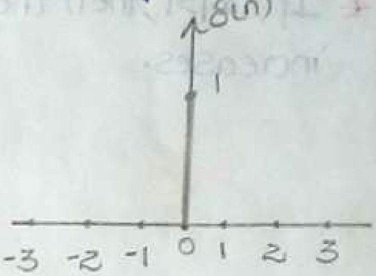
Signals in digital computers are represented by digital words or codes. The information carried by the digital code is generally in the form of discrete bits (logic pulses of "0" or "1") coded in a serial or parallel format.

Basic Discrete time signals :

1. Unit Impulse sequence :

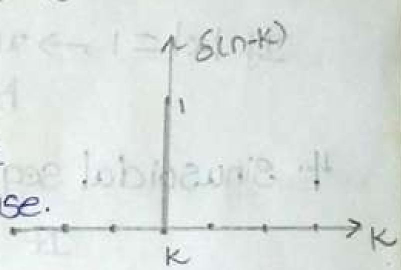
Unit impulse sequence contains only one non-zero element & is defined by

$$\delta(n) = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



Delayed unit impulse signal is denoted by $\delta(n-K)$. It has non zero element at sample time, K.

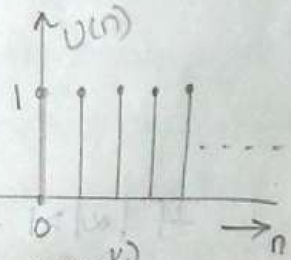
$$\delta(n-K) = \begin{cases} 1 & \text{for } n=K \\ 0 & \text{otherwise.} \end{cases}$$



2. Unit step sequence :

It is defined as

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Delayed unit step sequence is

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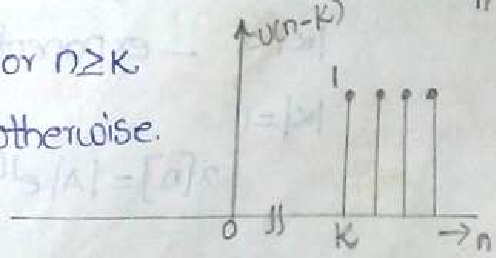
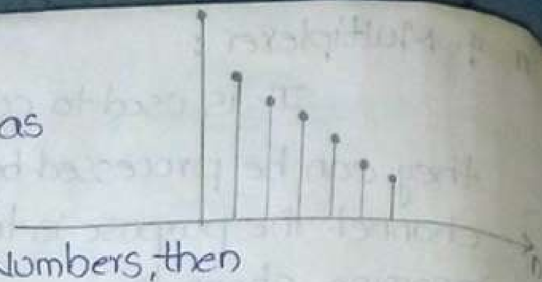


$$\delta(n) = u(n) - u(n-1).$$

12 3. Exponential sequence :

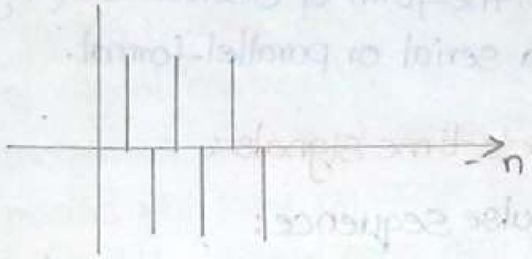
* It can be defined as

$$x(n) = A\alpha^n$$

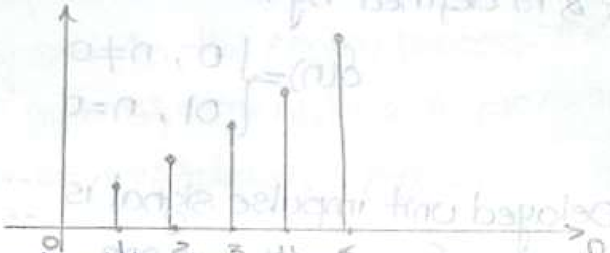


If A & α are real Numbers, then the sequence is real. If $0<\alpha<1$ and A is positive, then the sequence values are +ve & decreases with increasing n.

* for $-1<\alpha<0$, the sequence values alternate in sign but again decreases in Magnitude with increasing n.



* If $|\alpha|>1$, then the sequence grows in Magnitude as "n" increases.



If $|\alpha|=1 \rightarrow x(n)$ either is constant $(\alpha=1)$ or alternates between 1 & -1 $(\alpha=-1)$

4. Sinusoidal sequence :

It is defined as ,

$$x[n] = Ae^{j(\omega_0 n+\phi)}, \quad A>0$$

$$= |A|\,|\alpha|^n e^{j(\omega_0 n+\phi)}$$

$$= |A|\,|\alpha|^n \cos(\omega_0 n+\phi) + j|A|\,|\alpha|^n \sin(\omega_0 n+\phi)$$

If $|\alpha|>1$ – exponentially growing envelope.

$|\alpha|<1$ – exponentially Decaying envelope.

$|\alpha|=1$

$$x[n] = |A|e^{j(\omega_0 n+\phi)} = |A|\cos(\omega_0 n+\phi) + j|A|\sin(\omega_0 n+\phi)$$

Data conversion and quantization:

In D/A and A/D conversions, the MSB, LSB and weight of each digitally coded word are important in the conversion process.

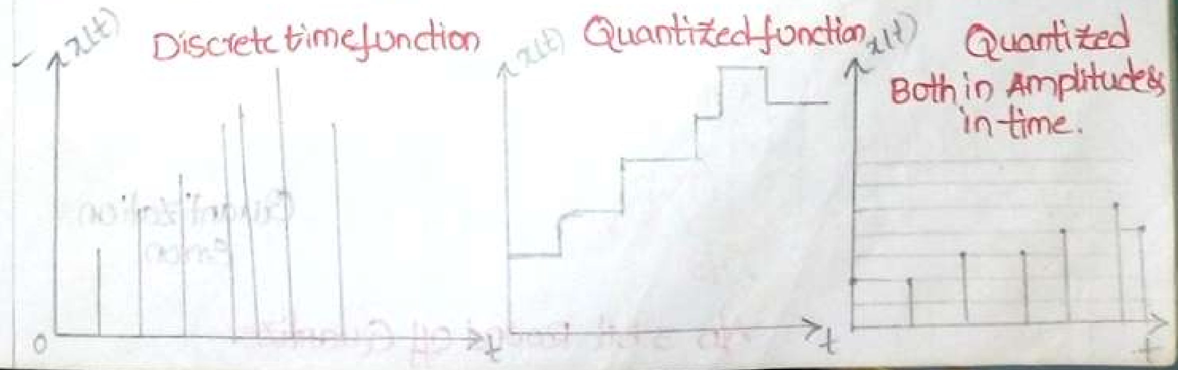
The practical A/D or D/A converters are based on the Natural binary code make use of the fractional code as shown in the below table.

	Binary Fraction (4 2 1)	Decimal Fraction	MSB $\frac{1}{2}$	Binary Fraction code, $\frac{1}{4}$	LSB $\frac{1}{8}$
0	000	0	0	0	0
1	001	$\frac{1}{8}$	0	0	1
2	010	$\frac{1}{4}$	0	1	0
3	011	$\frac{3}{8}$	0	1	1
4	100	$\frac{1}{2}$	1	0	0
5	101	$\frac{5}{8}$	1	0	1
6	110	$\frac{3}{4}$	1	1	0
7	111	$\frac{7}{8}$	1	1	1

*Quantization:

Digital computer in a control system requires the use of D/A & A/D converters.

The conversion of Analog signal to the corresponding digital signal is an approximation because the Analog signal can take an infinite no. of values, where as the variety of different Numbers which can be formed by finite set of digits is limited this approximation process is called "Quantization".



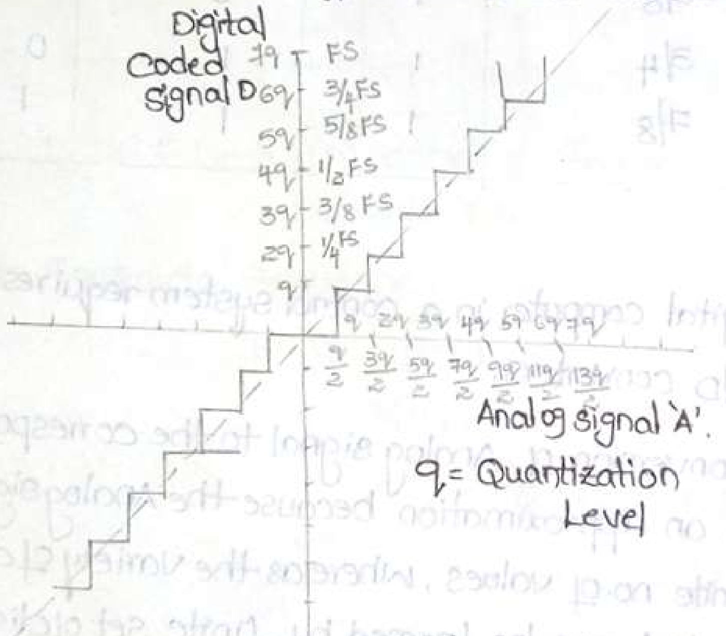
For a 3bit binary fraction code, the MSB has a weight of $\frac{1}{2}$ of full scale ($\frac{1}{2}$FS) & second bit has a weight of $\frac{1}{4}$FS & the LSB has a weight of $\frac{1}{8}$FS. [$2^{-3}FS = \frac{1}{2^3}FS = \frac{1}{8}FS$ Bcoz n=3]

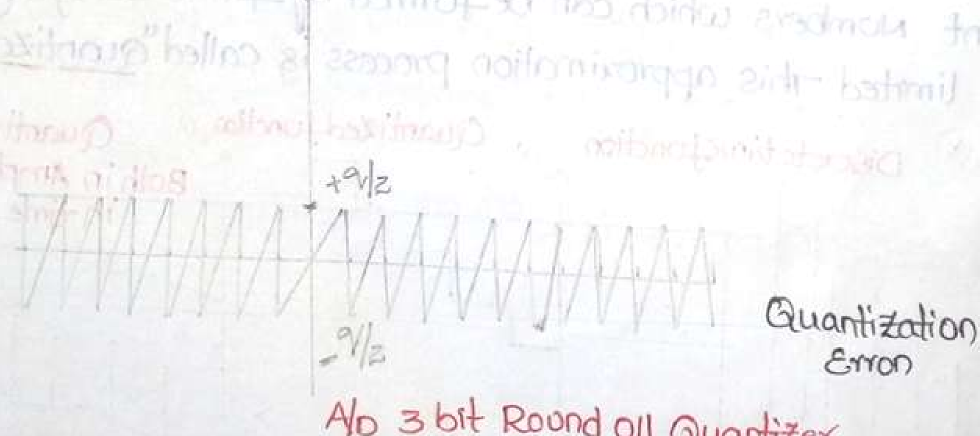
For an n-bit binary fractional code, the MSB still has a weight of $\frac{1}{2}$FS, but the LSB has a weight of 2^{-n}FS.

Regardless of whether integer or fractional coding is used n bit binary word defines 2^n distinct states, thus the word provides a resolution of one part in 2^n.

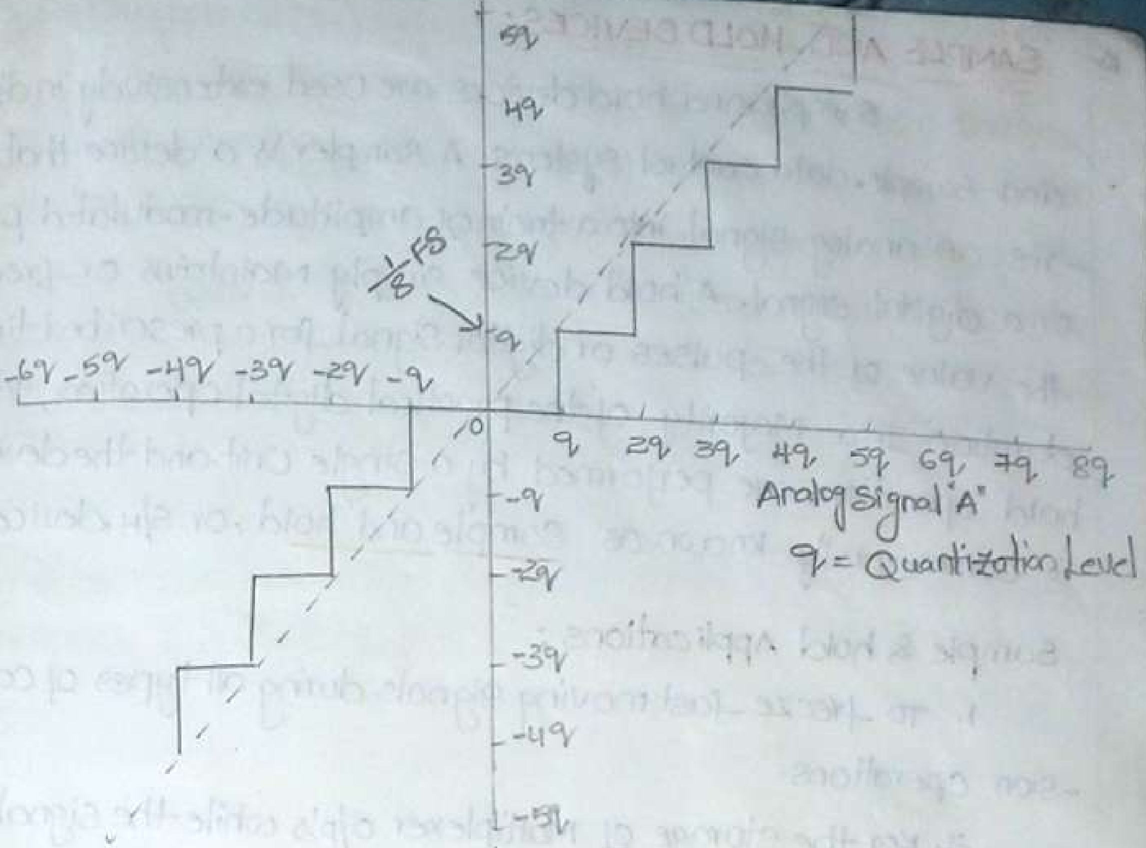
To improve the resolution by increasing the no. of bits, the same fullscale analog (or) digital signal value should be maintained.

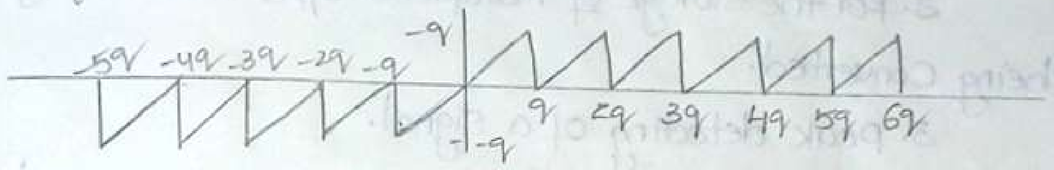
If the No of bits in the digital word is finite, only a finite resolution can be attained by A/D conversion. Since the digital o/p can assume only a finite no. of levels, the analog Number is rounded off or truncated into a digital number.





A/D 3 bit Round off Quantizer





A/D 3bit Truncation Quantizer

In rounded Quantization, the analog signal has decision levels at $\pm 0.5q, \pm 1.5q, \pm 2.5q \cdots \pm 6.5q$.

In truncated quantization, the analog signal has decision levels at $\pm q, \pm 2q, \pm 3q \cdots \pm 7q$.

The parameter q, which is equal to LSB is known as Quantization level.

The dotted line represents the ideal outputs if there is no Quantization.

The Difference between the straight line & the Quantization characteristics is the Quantization error.

The Maximum (digital) Error for truncated quantization is $\pm q$. & for round-off Quantization is $\pm 0.5q$.

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SAMPLE AND HOLD DEVICES:

Sample-and-hold devices are used extensively in digital and sample-data control systems. A sampler is a device that converts an analog signal into a train of amplitude-modulated pulses or a digital signal. A hold device simply Maintains or freezes the value of the pulses or digital signal for a prescribed time duration In a Majority of the practical digital operations, sample hold operations are performed by a single unit and the devices is commercially known as sample and hold, or S/H device.

Sample & hold Applications:

1. To freeze fast moving signals during all types of conversion operations.

2. For the storage of Multiplexer o/p's while the signal is being Converted.

3. Peak detecting of a signal.

A The typical S/H output is characterized by several sources of time delays and imperfect holding during the hold mode.

1. Acquisition Time (T_a):

When the sample command is given to the S/H device the unit does not begin to track the input signal instantaneously. It is the time measured from the instant the sample command is given to the time when the S/H o/p enters and remains within a specific error band around the input signal.

2. Aperature time (T_p):

When the hold command is given to the S/H device, while it is in the sample (track) mode, it will stay in this mode before reacting. The time between the Issuance of the hold command and the time the sampler is opened is called the aperature time. This delay is usually caused by the switching circuit time delay within the S/H.

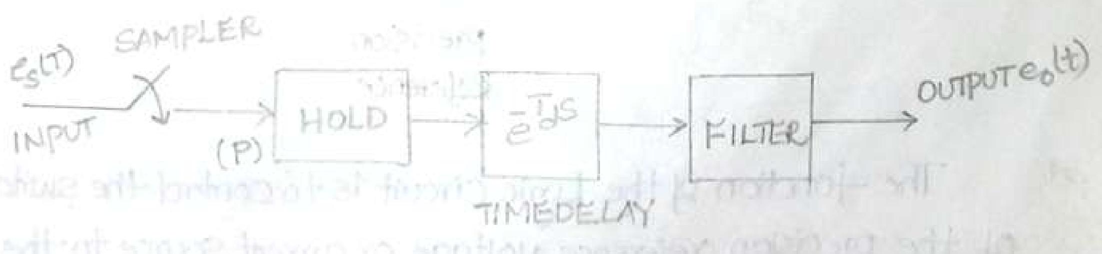
17 3 Settling Time (T_S):

In switching from the Sample mode to the hold mode, transient caused by capacitance feedthrough from the digital Logic Circuitry through the electronic switch to the analog signal path can occur. The time required for the transient oscillation to settle to within a certain percent of FS is called the settling time.

BLOCK DIAGRAM REPRESENTATION OF THE S/H DEVICE:

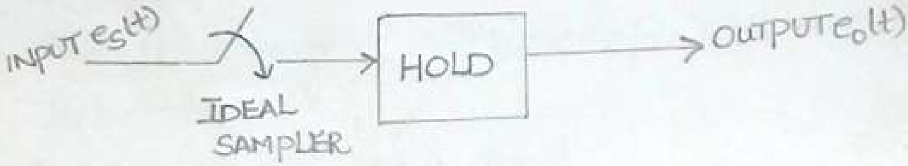
Although the S/H device is available electronically as a single chip, for Analytical purposes it is more convenient to treat the sampling & holding operations Separately.

A block diagram Approximation of the sample & hold Device



The sampler which can be regarded as pulse-Amplitude modulator, has a sampling period 'T' and a sampling duration 'p's. the hold devices simply holds the signal during the holding periods. Time delay approximates Acquistion time and the operature time delays. filter is used to represent finite time constant & dynamics of the buffer Amplifiers.

The Above sampler has a finite sampling duration p. In practice $p \ll T$ & time delay due to sampling and holding is small. In this case the sampler is called an ideal sampler. Since it is assumed to have a zero sampling duration i.e. $p=0$

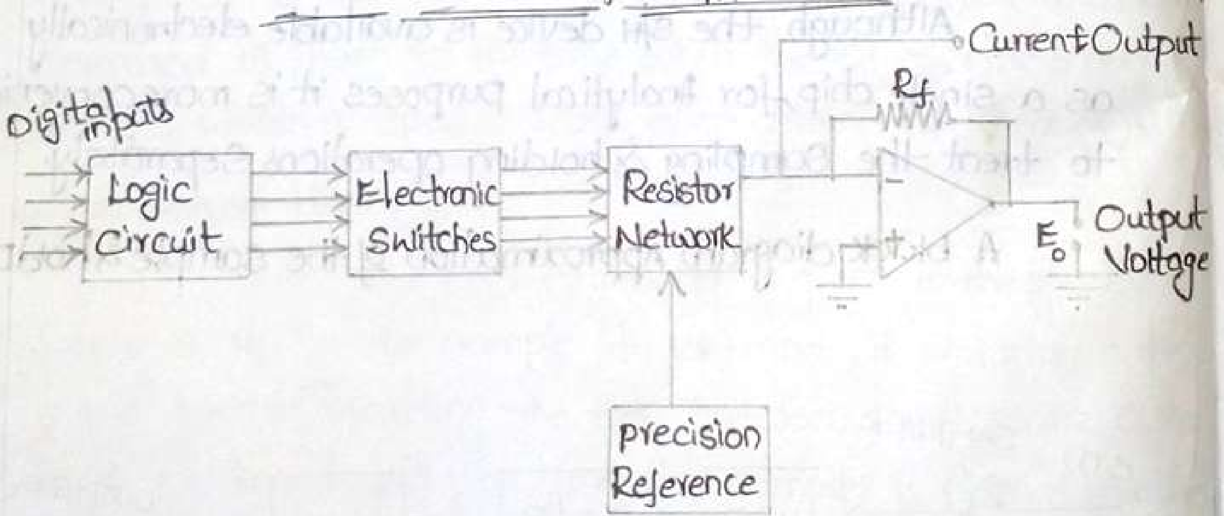


An ideal sample-Hold device

* DIGITAL-TO-ANALOG (D/A) CONVERSION:

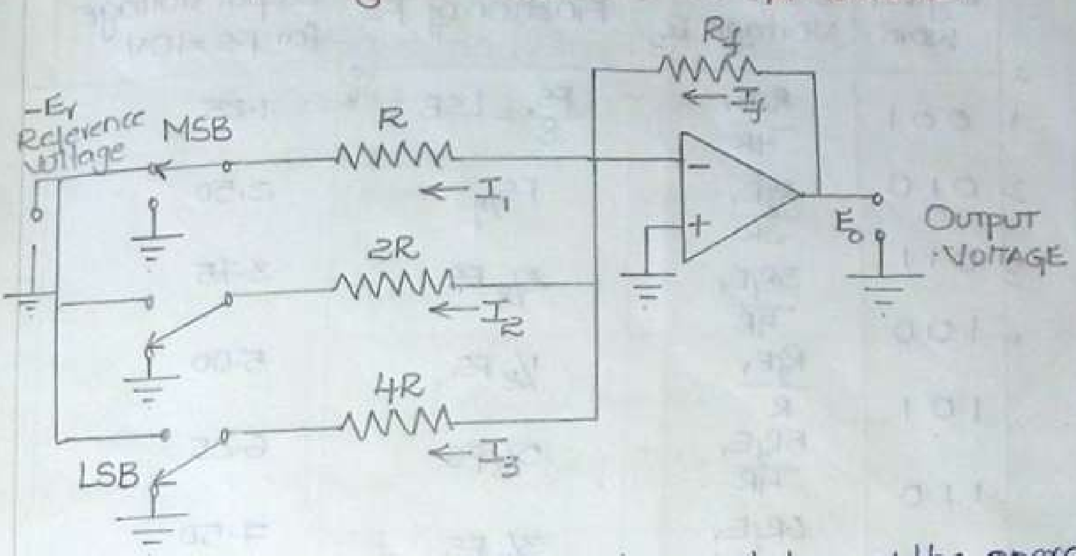
Digital-to-Analog conversion, or simply decoding, consists of transforming the Numerical information contained in a digitally coded word into an equivalent analog signal

Basic elements of a D/A Converter



The function of the Logic Circuit is to control the switching of the precision reference voltage or current source to the proper i/p terminals of the resistor Network as a function of (resistor) digital value of each digit input bit.

A Weighted-Resistor 3-bit D/A Converter



The values of the summing ressistors of the operational amplifier are weighted in a binary-fashion. Each of these resisto-rs is connected through an electronic switch to the reference voltage or to ground.

1 → appears at the control Logic circuit of a switch it closes the switch.

0 → connects the resistance to Ground [opens the switch]

Example: If the MSB branch is connected to the reference voltage $-E_r$, & other 2 switches are connected to Ground, this correspon-ding to a digital word of "100", the o/p voltage E_0 is.

$$E_0 = R_f I_f \longrightarrow (1)$$

since $I_f = I_1$ & $I_1 = E_r/R$

$$E_0 = \frac{R_f E_r}{R} \longrightarrow (2)$$

If the digital word is "110" then

$$E_0 = \left[\frac{1}{R} + \frac{1}{2R}\right] R_f E_r$$

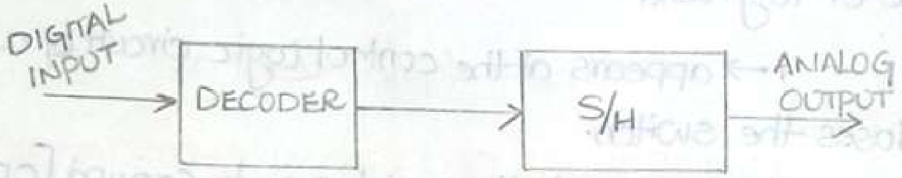
$$E_0 = \frac{3}{2R} R_f E_r \longrightarrow (3)$$

If the digital word is "111" then

$$E_0 = \left[\frac{1}{R} + \frac{1}{2R} + \frac{1}{4R}\right] R_f E_r = \frac{7}{4R} R_f E_r \longrightarrow (4)$$

Output voltages of the D/A of above figure

	Digital word	Output voltage E_0	Fraction of FS	output voltage for FS = 10V
0				
1	001	$\frac{R_f E_r}{4R}$	$\frac{FS}{8} = LSB = \frac{10}{8}$	1.25
2	010	$\frac{R_f E_r}{2R}$	$FS/4$	2.50
3	011	$\frac{3R_f E_r}{4R}$	$3/8\ FS$	3.75
4	100	$\frac{R_f E_r}{R}$	$1/2\ FS$	5.00
	101	$\frac{5R_f E_r}{4R}$	$5/8\ FS$	6.25
	110	$\frac{6R_f E_r}{4R}$	$3/4\ FS$	7.50
7	111	$\frac{7R_f E_r}{4R}$	$FS - LSB = 7/8\ FS$	8.75

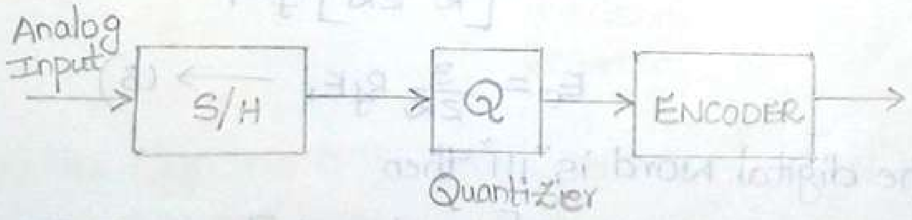


Block Diagram Representation of D/A Converter

* ANALOG-TO-DIGITAL CONVERSION :

Analog-to-Digital Conversion or simply encoding consists of converting the numerical information contained in an analog signal into a digitally coded word.

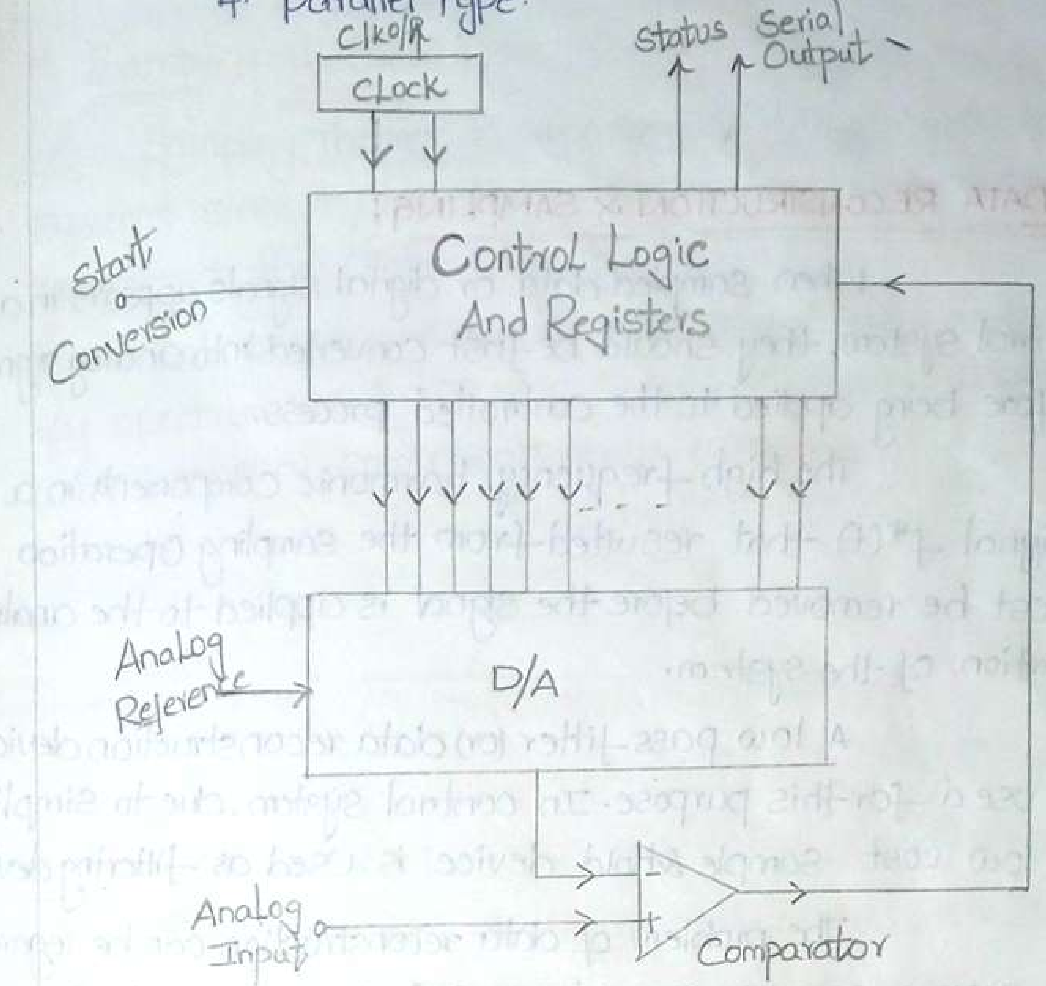
The sampling operation is needed to sample the analog signal at fixed periodic intervals. The input to an A/D converter is usually in the form of a voltage or a current reading that is quantized during the conversion process.



Block Diagram Representation of an A/D Converter

The most commonly used A/D converters are

1. Successive Approximation
2. Integration (Single & Dual slope)
3. Counter or Servo Type
4. parallel Type.



To illustrate the basic A/D conversion process, the successive-approximation type of A/D is briefly described. It consists of D/A, comparator & some control logic.

At the start of conversion, all the bits of the output of the A/D converter are set to zero (clearing), and then MSB is then set to one. The MSB, representing one half of full scale is then converted by the D/A converter internally and compared with the analog input.

If the input is greater than the converted MSB, then MSB=1 is left on. otherwise it is set to zero. The next significant bit is then turned on and compared and set, then a status line indicates that comparision is completed and the

* DATA RECONSTRUCTION & SAMPLING:

When sampled data or digital signals appears in a control system, they should be first converted into analog signals before being applied to the controlled process.

The high frequency harmonic components in a signal $f^*(t)$ that resulted from the sampling operation must be removed before the signal is applied to the analog portion of the system.

A low pass filter (or) data reconstruction device is used for this purpose. In control system, due to simplicity & low cost sample & hold device is used as filtering device

The problem of data reconstruction can be regarded as given a sequence of numbers, $f(0), f(T), f(2T) \cdots f(KT)$. a continuous signal $f(t)$, $t \geq 0$ is to be reconstructed from the information contained in the sequence, it is known as "extrapolation process".

The well known method of generating is based on the power series expansion of $f(t)$ in the interval between the sampling instants KT & $(K+1)T$.

$$f_K(t) = f(KT) + f'(KT)(t-KT) + \frac{f''(KT)}{2!}(t-KT)^2 + \cdots \rightarrow (1)$$

Where $f_K(t) = f(t)$ for $KT \leq t \leq (K+1)T$

$$f^n(KT) = \left.\frac{d^n f(t)}{dt}\right|_{t=KT}$$

To evaluate the coefficients of power series can be determined by obtaining derivative part of $f(t)$

$$f'(KT) = \frac{1}{T}\left[f(KT) - f(K-1)T\right]$$

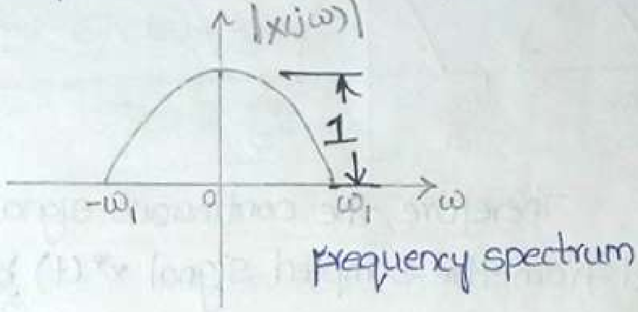
$$f''(KT) = \frac{1}{T}\left[f'(KT) - f'(K-1)T\right]$$

$$f'[(K-1)T] = \frac{1}{T}\left[f((K-1)T) - f((K-2)T)\right]$$

* Sampling Theorem :

Sampling Theorem is important in designing discrete time systems since it gives the minimum sampling frequency to reconstruct the original signal.

Let us assume a continuous signal $x(t)$ has the frequency spectrum as shown. This signal $x(t)$ does not contain any frequency components above ω_1 rad/sec.



Frequency spectrum

Statement : Let $x(t)$ be a band limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_1$. Then $x(t)$ is uniquely determined from its sample $x(K) = x(KT)$ if the sampling frequency $\omega_S = 2\pi/T$ is greater than $2\omega_1$ i.e., the sampling frequency must be atleast twice the highest frequency present in the signal.

Let us define,

$$X(S) = L[x(t)]$$

The Laplace transform of the sampled signal $x^*(t)$ is given by

$$X^*(S) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X(S + j\omega_S K)$$

$$S = j\omega$$

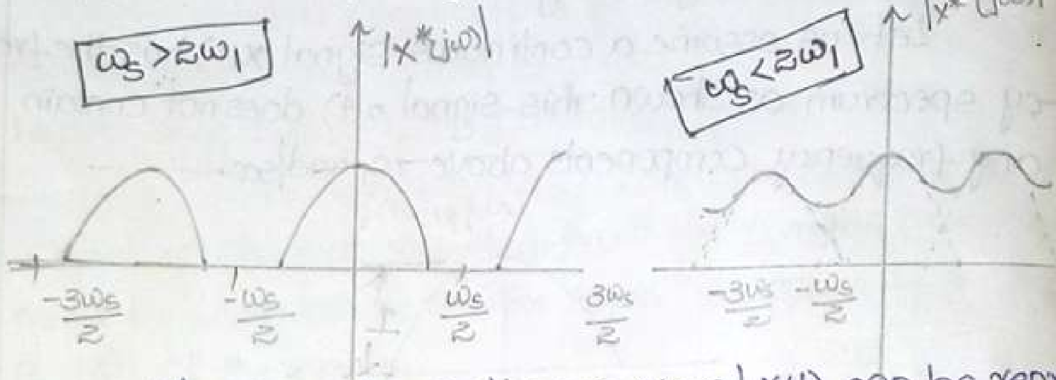
$$|X^*(j\omega)| = \frac{1}{Z}\sum_{K=-\infty}^{\infty} X[j\omega + j\omega_S K]$$

$$= \frac{1}{Z}\sum_{K=-\infty}^{\infty} X[j(\omega + \omega_S K)]$$

In the frequency, the component $|X(j\omega)|/T$ is called the primary component & ~~pr~~ other components are called complementary components.

If $\omega_s > 2\omega_1$, no two components of $|X^*(j\omega)|$ will overlap. Thus the original shape of $|X(j\omega)|$ is preserved by the sampling process.

If $\omega_s < 2\omega_1$, then the original shape of $|X(j\omega)|$ no longer appears in the plot of $|X^*(j\omega)|$ versus ω.



Therefore, the continuous signal $x(t)$ can be reproduced from the sampled signal $x^*(t)$ by filtering if & only if $\omega_s > 2\omega_1$, or $T < \pi/\omega_1$.

Hence, if sampling period T is smaller than π/ω_1, then the continuous signal can be reconstructed by use of lowpass filter, after signal has been sampled.

* THE ZERO-ORDER HOLD: [ZOH]

The sampling operation produces an amplitude modulated pulse signal. The function of the hold operation is to reconstruct the analog signal that has been transmitted as a train of pulse samples.

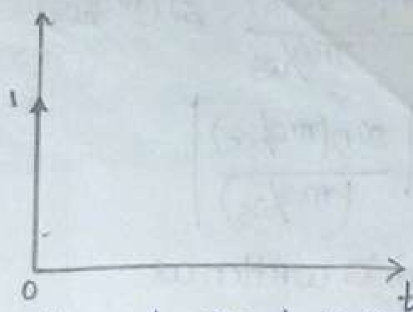
from equation ①

$$f_k(t) = f(kT)$$

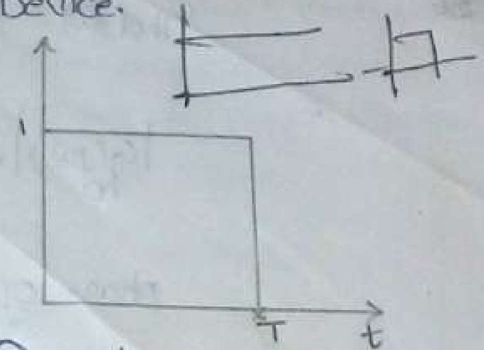
The device that performs the extrapolations is known as "ZERO ORDER HOLD".

since the polynomial is used is of the zeroth order.

Zero order hold is a Linear Device.



ⓐ Unit Impulse Input to ZOH



ⓑ Impulse Response of ZOH

Impulse response can be expressed as

$$g_{ho}(t) = U_s(t) - U_s(t-T)$$

$U_s(t)$ – Unit step function

taking Laplace transforms

$$G_{ho}(s) = \frac{1}{s} - \frac{e^{-Ts}}{s}$$

$$\boxed{G_{h0}(s) = \frac{1-e^{-sT}}{s}}$$

* **Frequency-Domain Characteristics of the ZOH :**

ZOH is a data-reconstruction or filtering Device, it is of intreset to examine its frequency-domain characteristics.

$$G_{ho}(s) = \frac{1-e^{-sT}}{s}$$

put $s=j\omega$

$$G_{ho}(j\omega) = \frac{1-e^{-j\omega T}}{j\omega}$$

$$G_{ho}(j\omega) = \frac{2e^{-j\omega T/2}\left(e^{j\omega T/2}-e^{-j\omega T/2}\right)}{j2\omega}$$

$$\frac{e^{j\omega T/2}-e^{-j\omega T/2}}{2j} = \sin\omega T/2$$

$$G_{ho}(j\omega) = \frac{2\sin(\omega T/2)}{\omega} \cdot e^{-j\omega T/2}$$

Where $T=$ sampling period in seconds

$$T = 2\pi/\omega_s$$

Where ω_s is the sampling frequency in rad/sec.

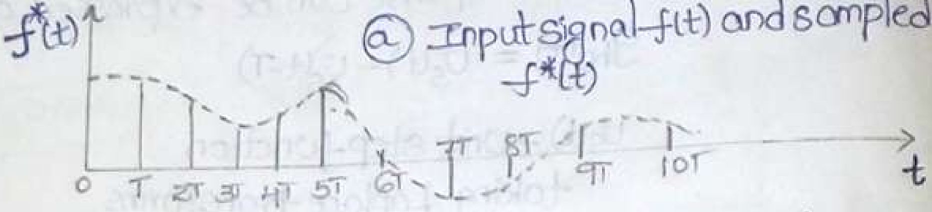
$$G_{ho}(j\omega) = \frac{2\pi}{\omega_s}\,\frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s}\,e^{-j\omega\left(\frac{2\pi/\omega_s}{2}\right)}$$

$$G_{ho}(j\omega) = \frac{2\pi}{\omega_s} \cdot \frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s} \cdot e^{-j(\pi\omega/\omega_s)}$$

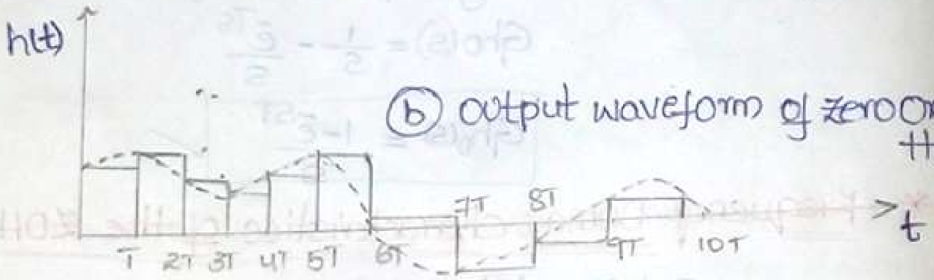
$$|G_{ho}(j\omega)| = \frac{2\pi}{\omega_s} \left| \frac{\sin(\pi\omega/\omega_s)}{(\pi\omega/\omega_s)} \right|$$

Phase of $G_{ho}(j\omega)$ is written as

$$\angle G(j\omega_s) = \angle \sin(\pi\omega/\omega_s) - \pi\omega/\omega_s \text{ rad}$$

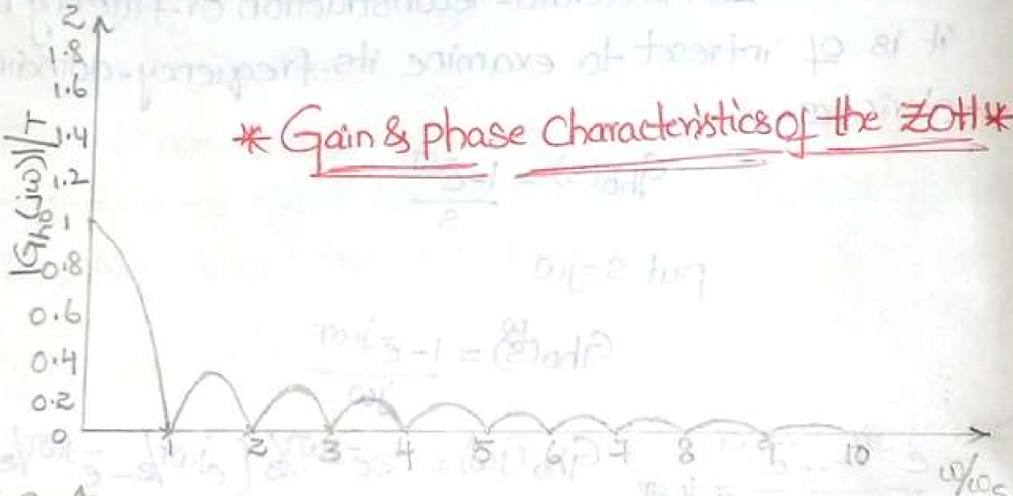


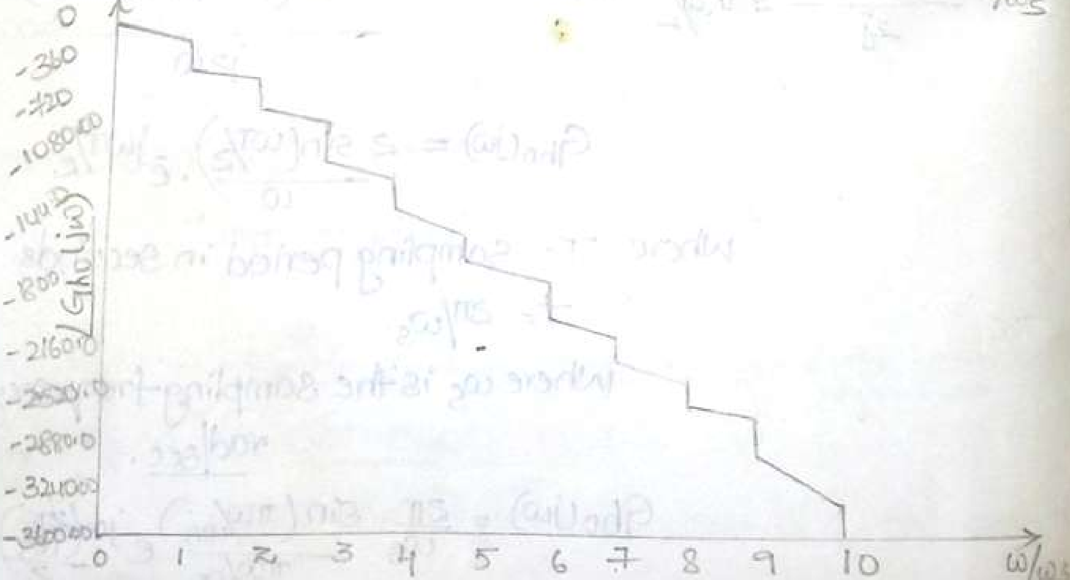
ⓐ Input signal f(t) and sampled signal $f^*(t)$



ⓑ output waveform of zero order Hold.

* Gain & phase Characteristics of the ZOH *





27 **Data Hold**: Data Hold is a process of generating a continuous-time signal $h(t)$ from a discrete time sequence $x(KT)$. The signal $h(t)$ during the time interval $KT \leq t \leq (K+1)T$ may be approximated by a polynomial in T as follows:

$$h(KT+T) = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_1 T + a_0$$

If $n=0$ Zero Order Hold ✓

UNIT-II

* Z-TRANSFORMATIONS *

A mathematical tool commonly used for the analysis and synthesis of discrete-time control systems is the Z-Transform. The role of the Z-Transform in discrete-time systems is similar to that of the Laplace transforms in continuous-time systems.

With the Z-Transform method, the solutions to linear difference equations become algebraic in Nature. The Z-Transformation transforms linear time invariant difference equations into algebraic equations in 'Z'.

Z-Transform :

Consider a time-function $x(t)$ with sampled values $x(0), x(T), x(2T) \ldots$

Where T - sampling period.

Z-Transform of the time function $x(t)$ or a sequence $x(KT)$ is defined as.

$$X(Z) = Z\left[x(t)\right] = Z\left[x(KT)\right] = \sum_{K=0}^{\infty} x(KT) Z^{-K}$$

If the Z-Transform is defined for the interval, $-\infty<t<\infty$, then it is referred to as 2 sided Z-Transform that is.

$$X(Z) = Z\left[x(t)\right] = \sum_{K=-\infty}^{\infty} x(KT) Z^{-K}$$

$$X(Z) = Z\left[x(K)\right] = \sum_{K=-\infty}^{\infty} x(K) Z^{-K}$$

Where $x(t) \neq 0$, for $t<0$

$x(K) \neq 0$, for $K<0$

* Realization of Z-Transform from Laplace Transform :

Let the output of an ideal sampler be $x^*(t)$ which is given by

$$x^*(t) = \sum_{K=0}^{\infty} x(KT)\delta(t-KT) = x(t)\delta_T(t) \rightarrow (1)$$

The Laplace Transform of $x^*(t)$ is

$$L\left[x^*(t)\right] = X^*(S) = \sum_{K=0}^{\infty} x(KT)e^{-KTS} \rightarrow (2)$$

Let us define, $z = e^{ST} \rightarrow (3)$

$S = \frac{1}{T}\ln z \rightarrow (4)$

where T = Sampling period

z = Complex Variable

with $S = \sigma + j\omega$

substitute (3),(4) in (2)

$$X^*\left(\frac{1}{T}\ln z\right) = X(z) = \sum_{K=0}^{\infty} x(KT)z^{-K}$$

The z-Transform of a continuous time signal canbe expanded as

$$X(z) = x(0) + x(T)z^{-1} + x(2T)z^{-2} + \cdots + x(KT)z^{-K} + \cdots$$

* z-Transforms of elementary functions :—

By considering one sided z-Transform.

1) Unit step function :

It is defined as

$$x(t) = \begin{cases} 1(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

z-Transform definition

$$X(z) = z\left[x(t)\right] = \sum_{K=0}^{\infty} x(t)z^{-K}$$

$$= \sum_{K=0}^{\infty} 1(t)z^{-K}$$

$$= \sum_{K=0}^{\infty} z^{-K}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$

$$= \frac{1}{1-z^{-1}}$$

$$\boxed{X(z) = \frac{z}{z-1}}$$

33 2) Unit ramp function:

Defined as,

$$x(t)=\begin{cases} t, & t\ge 0 \\ 0, & t<0 \end{cases} \quad \text{(or)} \quad x(KT)=KT,\ K=0,1,2\cdots$$

$$X(z)=Z[x(KT)]=\sum_{K=0}^{\infty} x(KT)z^{-K}$$

$$X(z)=Z[t]=Z[KT]$$

$$=\sum_{K=0}^{\infty} KTz^{-K}$$

$$=T\sum_{K=0}^{\infty} K\cdot z^{-K}$$

$$X(z)=T\left[z^{-1}+2z^{-2}+3z^{-3}+\cdots\right] \longrightarrow ①$$

Multiplying bothsides by z^{-1}

$$z^{-1}X(z)=T\left[z^{-2}+2z^{-3}+3z^{-4}+\cdots\right] \longrightarrow ②$$

① - ②

$$X(z)-z^{-1}X(z)=T\left[(z^{-1}+2z^{-2}+3z^{-3}+\cdots)-(z^{-2}+2z^{-3}+3z^{-4}+\cdots)\right]$$

$$(1-z^{-1})X(z)=T\left[z^{-1}+z^{-2}+z^{-3}+\cdots\right] \longrightarrow ③$$

Again Multiply z^{-1} on both sides

$$(z^{-1}-z^{-2})X(z)=T\left[z^{-2}+z^{-3}+\cdots\right] \longrightarrow ④$$

③ - ④

$$X(z)\left[1-2z^{-1}+z^{-2}\right]=T\left[z^{-1}+\cancel{z^{-2}}+\cancel{z^{-3}}+\cdots-\cancel{z^{-2}}-\cancel{z^{-3}}-\cdots\right]$$

$$X(z)\left[1-z^{-1}\right]^2=T\cdot z^{-1}$$

$$X(z)=\frac{T\cdot z^{-1}}{(1-z^{-1})^2}$$

$$=\frac{T\cdot(1/z)}{\left(\frac{z-1}{z}\right)^2}$$

$$\boxed{X(z)=T\frac{z}{(z-1)^2}}$$

3) Polynomial function a^k:

$$x(t)=x(k)=\begin{cases} a^k, & k=0,1,2 \\ 0, & k=0 \end{cases}$$

where a is constant

$$X(z) = Z[x(k)] = \sum_{k=0}^{\infty} x(k)\, z^{-k}$$

$$Z[a^k] = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} (az^{-1})^k$$

$$X(z) = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots \longrightarrow ①$$

Multiplying bothsides by az^{-1}

$$az^{-1}X(z) = az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots \longrightarrow ②$$

① - ②

$$[1 - az^{-1}]X(z) = [1 + \cancel{az^{-1}} + a^2z^{-2} + a^3z^{-3} \cdots] - [\cancel{az^{-1}} + a^2z^{-2} + a^3z^{-3} + \cdots]$$

$$(1 - az^{-1})X(z) = 1$$

$$X(z) = \frac{1}{1 - az^{-1}}$$

$$\boxed{X(z) = \frac{z}{z-a}}$$ ✓

4) Exponential Function :

$$x(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

let $t = kT$, $x(kT) = \begin{cases} e^{-akT}, & k \geq 0 \\ 0, & k < 0 \end{cases}$

let $X(z) = Z[x(kT)] = \sum_{k=0}^{\infty} x(kT)\, z^{-k}$

$$= Z[x(kT)] = \sum_{k=0}^{\infty} e^{-akT} z^{-k}$$

$$X(z) = 1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + \cdots$$

Multiplying bothsides by $e^{-aT}z^{-1}$ & subtracting

$$X(z)e^{-aT} = e^{-aT}[1 + e^{-aT}z^{-1} + ze^{-2aT}z^{-2} + \cdots]$$

$$X(z)e^{-aT} = e^{-aT} + e^{(-aT)^2}z^{-1}$$

$$\boxed{X(z) = \frac{1}{1 - e^{-aT}z^{-1}} = \frac{z}{z - e^{-aT}}}$$ ✓

35 5) Sinusoidal function:

$$x(t) = \begin{cases} \sin\omega t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

We know that. $e^{j\omega t} = \cos\omega t + j\sin\omega t$

$e^{-j\omega t} = \cos\omega t - j\sin\omega t$

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

since the Z-Transformation of exponential function is

$$Z[e^{-at}] = \frac{1}{1-e^{-aT}z^{-1}}$$

$$X(Z) = Z[\sin\omega t] = Z\left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right]$$

$$= \frac{1}{2j}\left[Z(e^{j\omega t} - e^{-j\omega t})\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1-e^{j\omega T}z^{-1}} - \frac{1}{1-e^{-j\omega T}z^{-1}}\right]$$

$$= \frac{1}{2j}\left[\frac{\cancel{1} - e^{-j\omega T}z^{-1} - \cancel{1} + e^{j\omega T}z^{-1}}{(1-e^{+j\omega T}z^{-1})(1-e^{-j\omega T}z^{-1})}\right]$$

$$= \frac{1}{2j}\left[\frac{(e^{j\omega T}z^{-1} - e^{-j\omega T}z^{-1})}{1 - e^{-j\omega T}z^{-1} - e^{j\omega T}z^{-1} + e^{j\omega T}e^{-j\omega T}z^{-2}}\right]$$

$$= \frac{1}{2j}\left[\frac{z^{-1}(e^{j\omega T} - e^{-j\omega T})}{1 - z^{-1}(e^{j\omega T} + e^{-j\omega T}) + z^{-2}}\right]$$

$$= \frac{z^{-1}\sin\omega T}{1 - z^{-1}2\cos\omega T + z^{-2}}$$

$$\boxed{Z[\sin\omega t] = \frac{z\sin\omega T}{z^2 + z\,2\cos\omega T + 1}}$$

Pb1) Obtain the Z-Transform of the Cosine function

$$x(t) = \begin{cases} \cos\omega t, & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Sol Let $t = KT$

$$x(KT) = \begin{cases} \cos\omega KT & K \geq 0 \\ 0 & K < 0 \end{cases}$$

$$\boxed{X(z) = z[x(KT)] = \sum_{K=0}^{\infty} x(KT) z^{-K}}$$

we know that

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\& \quad z[e^{-at}] = \frac{1}{1-e^{-aT}z^{-1}}$$

$$X(z) = z\left[\frac{e^{j\omega t}+e^{-j\omega t}}{2}\right] = \frac{1}{2}z\left(e^{j\omega t}+e^{-j\omega t}\right)$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j\omega T}z^{-1}} + \frac{1}{1-e^{-j\omega T}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{(1-e^{-j\omega T}z^{-1} + 1 - e^{j\omega T}z^{-1})}{(1-e^{j\omega T}z^{-1})(1-e^{-j\omega T}z^{-1})}\right]$$

$$= \frac{1}{2}\left[\frac{2-(e^{j\omega T}+e^{-j\omega T})z^{-1}}{1-z^{-1}(e^{j\omega T}+e^{-j\omega T})+z^{-2}}\right]$$

$$= \frac{1-z^{-1}\cos\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}$$

$$\boxed{z[\cos\omega t] = \frac{z^2 - z\cos\omega T}{z^2-2z\cos\omega T+1}}$$

2) Obtain the z-Transform of

$$X(S) = \frac{1}{S(S+1)}$$

Sol

$$S \to t \to z$$

Inv. Laplace $\quad z$

$$X(S) = \frac{1}{S(S+1)}$$

$$L^{-1}[X(S)] = L^{-1}\left[\frac{1}{S(S+1)}\right] = L^{-1}\left[\frac{1}{S} - \frac{1}{S+1}\right]$$

$$\frac{1}{S(S+1)} = \frac{A}{S} + \frac{B}{(S+1)}$$

$$\frac{1}{S(S+1)} = \frac{A(S+1)+BS}{S(S+1)}$$

$$A(S+1)+BS = 1$$

$$A+B=0$$

$$A=1$$

$$1+B=0$$

$$B=-1$$

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$x(t) = 1-e^{-t} \quad t \geq 0$

$t = KT$

$x(KT) = 1-e^{-KT}$

$X(z) = z[x(KT)] = \sum_{K=0}^{\infty} x(KT) z^{-K}$

$z[1-e^{-KT}] = \sum_{K=0}^{\infty} (1-e^{-KT}) z^{-K}$

$= \sum_{K=0}^{\infty} z^{-K} - \sum_{K=0}^{\infty} e^{-KT} z^{-K}$

$= \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-T}z^{-1}}$

$= \frac{(1-e^{-T}z^{-1})-(1-z^{-1})}{(1-z^{-1})(1-e^{-T}z^{-1})}$

$= \frac{-e^{-T}z^{-1}+z^{-1}}{(1-z^{-1})(1-e^{-T}z^{-1})}$

$$\boxed{z[1-e^{-KT}] = \frac{(1-e^{-T})z^{-1}}{(1-z^{-1})(1-e^{-T}z^{-1})}}$$

3) Obtain the z-Transformation of K^2

Sol

$X(z) = z[x(K)] = \sum_{K=0}^{\infty} x(K) z^{-K}$

$z[K^2] = \sum_{K=0}^{\infty} K^2 z^{-K}$

$X(z) = 0 + 1^2 \cdot z^{-1} + (2)^2 z^{-2} + (3)^2 \cdot z^{-3} + \cdots$

$X(z) = z^{-1} + 4z^{-2} + 9z^{-3} + \cdots \longrightarrow (1)$

Multiplying bothsides by z^{-1}

$z^{-1} X(z) = z^{-2} + 4z^{-3} + 9z^{-4} + \cdots \longrightarrow (2)$

(1)-(2)

$(1-z^{-1}) X(z) = (z^{-1} + 4z^{-2} + 9z^{-3} + \cdots) - (z^{-2} + 4z^{-3} + 9z^{-4} + \cdots)$

$(1-z^{-1}) X(z) = z^{-1} + 3z^{-2} + 5z^{-3} + \cdots \longrightarrow (3)$

Again multiplying (3) by z^{-1}

$z^{-1}(1-z^{-1}) X(z) = z^{-2} + 3z^{-3} + 5z^{-4} + \cdots \longrightarrow (4)$

(3)-(4) $(1-z^{-1}) X(z) [1-z^{-1}] = (z^{-1} + 3z^{-2} + 5z^{-3} + \cdots) - (z^{-2} + 3z^{-3} + 5z^{-4} + \cdots)$

$$(1-z^{-1})^2 X(z) = z^{-1} + 2z^{-2} + 2z^{-3} + \cdots \longrightarrow (5)$$

Again Multiplying on both sides by z^{-1}

$$z^{-1}(1-z^{-1})^2 X(z) = z^{-2} + 2z^{-3} + 2z^{-4} + \cdots \longrightarrow (6)$$

(5)-(6)

$$(1-z^{-1})^2 X(z)[1-z^{-1}] = (z^{-1}+2z^{-2}+2z^{-3}+\cdots) - (z^{-2}+2z^{-3}+2z^{-4}\cdots)$$

$$= z^{-1} + z^{-2}$$

$$(1-z^{-1})^3 X(z) = z^{-1}(1+z^{-1})$$

$$X(z) = \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$$

$$= \frac{1/z\,(1+1/z)}{(1-1/z)^3}$$

$$Z(K^2) \Rightarrow \boxed{X(z) = \frac{z(z+1)}{(z-1)^3}}$$

4) Obtain the Z-Transform of $\frac{1}{a}(1-e^{-at})$, where a is constant

<u>Sol</u> let $t = KT$

$$\text{Def}: X(z) = Z[X(KT)] = \sum_{K=0}^{\infty} x(KT) z^{-K}$$

$$= \sum_{K=0}^{\infty} \frac{1}{a}(1-e^{-aKT}) z^{-K}$$

$$= \frac{1}{a}\sum_{K=0}^{\infty} (z^{-K} - e^{-aKT} z^{-K})$$

$$= \frac{1}{a}\left(\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}\right)$$

$$= \frac{1}{a}\left(\frac{\cancel{1}-e^{-aT}z^{-1}\cancel{-1}+z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}\right)$$

$$= \frac{1}{a}\left(\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}\right)$$

$$\boxed{X(z) = \frac{1}{a}\,\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}}$$

5) Obtain the Z-Transform of $t^2 e^{-at}$, where a is Constant

<u>Sol</u> let $t = KT$

$$X(z) = z[x(KT)] = \sum_{K=0}^{\infty} x(KT) z^{-K}$$

$$z[t^2 e^{-at}] = z[(KT)^2 e^{-aKT}]$$

$$= \sum_{K=0}^{\infty} (K^2 T^2 e^{-aKT}) z^{-K}$$

$$= T^2 \sum_{K=0}^{\infty} K^2 e^{-aKT} \cdot z^{-K}$$

$$= T^2 \sum_{K=0}^{\infty} e^{-aKT} (K^2 \cdot z^{-K})$$

We know that $z[K^2] = \dfrac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$

$$z[e^{-at}] = \frac{1}{1-e^{-aT} z^{-1}}$$

$$X(z) = z[T^2 K^2 z^{-K}] = T^2 [z(K^2 z^{-K})]$$

$$\boxed{X(z) = T^2 \left[\frac{e^{-aT} z^{-1} (1+e^{-aT} z^{-1})}{(1-e^{-aT} z^{-1})^3} \right]}$$

6) Obtain the z-Transform of t^2

sol let $t = KT$

$$X(z) = z[x(KT)] = \sum_{K=0}^{\infty} x(KT) z^{-K}$$

$$X(z) = z[t^2] = z[(KT)^2] = \sum_{K=0}^{\infty} K^2 T^2 z^{-K}$$

$$= T^2 \sum_{K=0}^{\infty} K^2 z^{-K}$$

We know that $z(K^2) = \dfrac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$

$$\boxed{z[K^2 T^2] = T^2 \left[\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3} \right]}$$

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Theorems of Z-Transform :

1) Multiplication by a Constant :

If $X(z)$ is the Z-Transform of $x(t)$, then

$$Z[a x(t)] = a Z[x(t)] = a X(z)$$

where 'a' is Constant.

Proof: $$Z[a x(t)] = \sum_{K=0}^{\infty} a x(KT) z^{-K}$$

$$= a \sum_{K=0}^{\infty} x(KT) z^{-K}$$

$$= a X(z)$$

2) Linearity :

If $f(K)$ & $g(K)$ are Z-Transformable and α & β are scalars, then $x(K)$ formed by a Linear Combination

$$x(K) = \alpha f(K) + \beta g(K)$$

has the Z-Transform

$$X(z) = \alpha F(z) + \beta G(z)$$

Proof: $$X(z) = Z[x(K)] = Z[\alpha f(K) + \beta g(K)]$$

$$= \sum_{K=0}^{\infty} [\alpha f(K) + \beta g(K)] z^{-K}$$

$$= \alpha \sum_{K=0} f(K) z^{-K} + \beta \sum_{K=0}^{\infty} g(K) z^{-K}$$

$$= \alpha Z[f(K)] + \beta Z[g(K)]$$

$$\boxed{X(z) = \alpha F(z) + \beta G(z)}$$

3) Multiplication by a^K :

If $X(z)$ is the Z transform of $x(K)$, then the Z-Transform of $a^K x(K)$ Can be

$$Z[a^K x(K)] = X(a^{-1} z)$$

iii) **proof** :-

$$Z[a^k x(k)] = \sum_{k=0}^{\infty} a^k x(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} (a^{-1}z)^{-k} x(k)$$

$$\boxed{Z[a^k x(k)] = X(a^{-1}z)}$$

4) Final Value Theorem :-

Suppose that $x(k)$, where $x(k)=0$ for $k<0$ has the Z-Transform $X(z)$ and that all the poles of $X(z)$ lies inside the unit circle, with the possible exception of a simple pole at $z=1$. Then the final value of $x(k)$ i.e., the value of $x(k)$ as $k\to\infty$, Can be

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1} \left[(1-z^{-1}) X(z)\right]$$

proof :

$$Z[x(k)] = X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} \longrightarrow ①$$

$$Z[x(k-1)] = z^{-1}X(z) = \sum_{k=0}^{\infty} x(k-1) z^{-k} \longrightarrow ②$$

① - ②

$$\sum_{k=0}^{\infty} x(k) z^{-k} - \sum_{k=0}^{\infty} x(k-1) z^{-k} = X(z) - z^{-1}X(z)$$

$$= X(z)\left[1-z^{-1}\right]$$

taking limit as $z\to 1$

$$\lim_{z\to 1} \left[\sum_{k=0}^{\infty} x(k) z^{-k} - \sum_{k=0}^{\infty} x(k-1) z^{-k}\right] = \lim_{z\to 1} \left[1-z^{-1}\right] X(z)$$

5) Initial Value Theorem :

If $x(t)$ has the Z-Transform $X(z)$ & if $\lim_{z\to\infty} X(z)$ exists, then the initial value $x(0)$ of $x(t)$ or $x(k)$ is given by

$$x(0) = \lim_{z\to\infty} X(z)$$

proof :

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

$$= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

if $z \to \infty$

$$\lim_{z\to\infty} X(z) = x(0) + \cancel{x(1)\frac{1}{\infty}}^{\,0} + \cancel{x(2)\frac{1}{\infty}}^{\,0} + \cdots$$

$$\boxed{\lim_{z\to\infty} X(z) = x(0)}$$

6) Shifting Theorem :

Also known as real Translation Theorem. if $x(t)$ for $t<0$ & $x(t)$ has the z-transform $X(z)$, then.

$$z[x(t-nT)] = z^{-n}X(z)$$

$$\& \ z[x(t+nT)] = z^{n}\left[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}\right]$$

where 'n' is +ve Integer (or) zero

Proof : $z[x(t-nT)] = \sum_{k=0}^{\infty} x(kT-nT)z^{-k}$ $\quad \boxed{t=kT}$

$$= \sum_{k=0}^{\infty} x[T(k-n)]z^{-k}$$

Multiply & Divide with z^{-n}

$$= \sum_{k=0}^{\infty} x[T(k-n)]z^{-k} * \frac{z^{-n}}{z^{-n}}$$

$$= z^{-n}\sum_{k=0}^{\infty} x[T(k-n)]\frac{z^{-k}}{z^{-n}}$$

$$= z^{-n}\sum_{k=0}^{\infty} x[T(k-n)]\, z^{-k+n}$$

$$= z^{-n}\sum_{k=0}^{\infty} x[T(k-n)]\, z^{-(k-n)}$$

$$\boxed{z[x(t-nT)] = z^{-n}X(z)}$$

$$z[x(t+nT)] = \sum_{k=0}^{\infty} x(kT+nT)z^{-k}$$

$$= z^{n}\sum_{k=0}^{\infty}$$

//

$$\boxed{Z[x(t+nT)] = z^{n}\left[X(z) - \sum_{K=0}^{n-1} x(KT)\, z^{-K}\right]}$$

From the above Equation

$$Z[x(K+1)] = zX(z) - zx(0)$$

$$Z[x(K+2)] = z^2X(z) - z^2x(0) - zx(1).$$

7) Complex translation Theorem:

If $x(t)$ has the z-transform $X(z)$, then the z-transform of $e^{-at}x(t)$ can be given by $X(z \cdot e^{aT})$.

proof: $$Z[e^{-at}x(t)] = \sum_{K=0}^{\infty} x(KT)\, e^{-aKT} z^{-K}$$

$$= \sum_{K=0}^{\infty} x(KT)\left(z \cdot e^{aT}\right)^{-K}$$

$$\boxed{Z[e^{-at}x(t)] = X(z \cdot e^{aT})}$$

Problems

$Z[e^{-at}\sin\omega t]$

$$Z[\sin\omega t] = \frac{z^{-1}\sin\omega T}{1 - 2z^{-1}\cos\omega T + z^{-2}}$$

Substituting $z = ze^{aT}$ according complex translati-on theorem,

$$Z[e^{-at}\sin\omega t] = \frac{z^{-1}e^{-aT}\sin\omega T}{1 - 2e^{-aT}z^{-1}\cos\omega T + e^{-2aT}z^{-2}}$$

Similarly

$$Z[e^{-at}\cos\omega t] = \frac{1 - e^{-aT}z^{-1}\cos\omega T}{1 - 2e^{-aT}z^{-1}\cos\omega T + e^{-2aT}z^{-2}}$$

$Z[t \cdot e^{-at}]$

We know $$Z(t) = \frac{Tz^{-1}}{(1-z^{-1})^2}$$

$$\therefore Z[t e^{-at}] = X[z \cdot e^{aT}] = \frac{T e^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}$$

44 Find Z Transform of unit step function, delayed by 1 Sampling period & 4 Sampling periods.

Using Shifting Theorem

$$Z[1(t-T)] = z^{-1} Z[1(t)] = z^{-1}\frac{1}{1-z^{-1}} = \frac{z^{-1}}{1-z^{-1}}$$

$$Z[1(t-4T)] = z^{-4} Z[1(t)] = z^{-4}\left(\frac{1}{1-z^{-1}}\right) = \frac{z^{-4}}{1-z^{-1}}$$

⟹ Obtain the Z-Transform of,

$$f(a) = \begin{cases} a^{k-1}, & k=1,2,3\ldots \\ 0, & k \leq 0 \end{cases}$$

Sol

$$Z[x(k-1)] = z^{-1}X(z)$$

$$Z[a^k] = \frac{1}{1-az^{-1}}$$

$$Z[f(a)] = Z[a^{k-1}]$$

$$\boxed{= z^{-1}\frac{1}{1-az^{-1}}}$$

8. Differentiation :

$$Z[nx(n)] = z^{-1}\frac{dX(z)}{dz} \text{ (or) } -z\frac{dX(z)}{dz}$$

9. Convolution :

$$Z[x_1(n) * x_2(n)] = X_1(z) \cdot X_2(z).$$

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INVERSE Z-TRANSFORMATION :

By using the Inverse Z-Transform, only the time sequence at the sampling instants is obtained. Thus, the inverse Z-Transform of $X(Z)$ yields a unique $x(K)$, but does not yield a unique $x(t)$

There are 3 methods to obtain the inverse Z-transform of a given function.

1. Direct Division Method
2. Partial Fraction Expansion Method
3. Inverse Integral Method.

1. Direct Division Method :

In Direct Division Method we obtain the inverse Z-Transform by expanding $X(Z)$ into an infinite power series in Z^{-1}.

Pb1. Find $x(K)$ for $K = 0, 1, 2, 3, 4$ when $X(Z)$ is given by

$$X(Z) = \frac{10Z+5}{(Z-1)(Z-0.2)}$$

Sol

$$X(Z) = \frac{10Z+5}{(Z-1)(Z-0.2)} = \frac{10Z+5}{Z^2-1.2Z+0.2}$$

$$= \frac{10Z+5}{Z^2\left[1-1.2Z^{-1}+0.2Z^{-2}\right]}$$

$$= \frac{10Z^{-1}+5Z^{-2}}{1-1.2Z^{-1}+0.2Z^{-2}}$$

$$1-1.2Z^{-1}+0.2Z^{-2} \;\big)\; 10Z^{-1}+5Z^{-2} \;\big(\; 10Z^{-1}+17Z^{-2}+18.4Z^{-3}$$

$$\cancel{10Z^{-1}} - 12Z^{-2} + 2Z^{-3}$$
$$- \quad + \quad -$$
$$17Z^{-2} - 2Z^{-3}$$
$$\cancel{17Z^{-2}} - 20.4Z^{-3} + 3.4Z^{-4}$$
$$- \quad + \quad$$
$$18.4Z^{-3} - 3.4Z^{-4}$$
$$\cancel{18.4Z^{-3}} - 22.08Z^{-4} + 3.68Z^{-5}$$
$$- \quad + \quad -$$

$$X(z) = 10z^{-1} + 17z^{-2} + 18.4z^{-3} + 18.68z^{-4} + \cdots$$

$x(0) = 0,$

$x(1) = 10$

$x(2) = 17$

$x(3) = 18.4$

$x(4) = 18.68$

2. Find $x(k)$ when $X(z) = \frac{1}{z+1}$

Sol

$$X(z) = \frac{1}{z+1} = \frac{z^{-1}}{1+z^{-1}}$$

$$= z^{-1} - z^{-2} + z^{-3} - z^{-4} + \cdots$$

by comparing this Infinite series expansion of $X(z)$ with $X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$, we obtain

$x(0) = 0,$

$x(1) = 1,\ x(2) = -1,\ x(3) = 1,\ x(4) = -1$

3. Find Inverse z-Transformation for the given function $1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$ and also calculate $x(0), x(4), x(5)$

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + \cdots$$

$x(0) = 1;\ x(1) = 2;\ x(2) = 3;\ x(3) = 4$

$x(4) = 0$

$x(5) = 0$

1. Direct Division

2. Partial Fraction Method :-

To find the inverse z transform, if $X(z)$ has one or more zeros at the origin $(z=0)$, then $X(z)/z$ (or) $X(z)$ is expanded into a sum of simple first (or) second order terms by partial fraction expansion, and a z transform of $x(t)$ (or) $x(kT)$ expanded into partial fractions are the functions of z appearing in tables of z-Transforms.

47 Pb1. Find $x(KT)$ if $X(z)$ is given by

$$X(z) = \frac{10z}{(z-1)(z-2)}$$

so)

$$X(z) = \frac{10z}{(z-1)(z-2)}$$

$$\frac{X(z)}{z} = \frac{10}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$\frac{10}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$\frac{10}{\cancel{(z-1)(z-2)}} = \frac{A(z-2)+B(z-1)}{\cancel{(z-1)(z-2)}}$$

$$10 = A(z-2)+B(z-1)$$

$$10 = z(A+B) - 2A - B$$

$$A+B = 0 \longrightarrow (1)$$

$$-2A-B = 10 \longrightarrow (2)$$

$$\begin{array}{r} A+\cancel{B} = 0 \\ -2A-\cancel{B} = 10 \\ \hline -A = 10 \end{array}$$

$$B = +10$$

$$\frac{X(z)}{z} = \frac{10}{(z-1)(z-2)} = \frac{-10}{(z-1)} + \frac{10}{(z-2)}$$

$$\frac{X(z)}{z} = \frac{-10z}{z-1} + \frac{10z}{z-2}$$

$$X(z) = \frac{-10}{1-z^{-1}} + 10\frac{1}{1-2z^{-1}}$$

$$z^{-1}\left[\frac{1}{1-z^{-1}}\right] = 1$$

$$z^{-1}\left[\frac{1}{1-2z^{-1}}\right] = 2^K$$

$$\therefore x(KT) = -10 + 2^K$$

Pb2. $X(z) = \frac{z(z+2)}{(z-1)^2}$

$$\frac{X(z)}{z} = \frac{(z+2)}{(z-1)^2}$$

$$\frac{(z+2)}{(z-1)^2} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2}$$

48. $\frac{X(z)}{z} = \frac{z+2}{(z-1)^2} = \frac{1}{(z-1)} + \frac{3}{(z-1)^2}$

$$\frac{z+2}{(z-1)^2} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2}$$

$$\frac{z+2}{\cancel{(z-1)^2}} = \frac{A(z-1)+B}{\cancel{(z-1)^2}}$$

$$z+2 = Az - A + B$$

$$\underline{A=1}$$
$$-A+B=2$$
$$-1+B=2$$
$$\underline{B=3}$$

$$X(z) = \frac{z}{z-1} + \frac{3z}{(z-1)^2}$$

$$z^{-1}\left[\frac{z}{z-1}\right] \text{ (or) } z^{-1}\left[\frac{1}{1-z^{-1}}\right] = 1$$

$$z^{-1}\left[\frac{z}{(z-1)^2}\right] = 3t$$

$$x(KT) = 1+3t$$

$$\boxed{x(KT) = 1+3KT}$$

$$x(0) \Rightarrow K=0 \Rightarrow 1+3(0)T = 1$$
$$x(1) \Rightarrow K=1 \Rightarrow 1+3(1)T = 1+3T$$

Pb3. $X(z) = \frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$ find Inverse z-Transform.

sol

$$X(z) = \frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$$

$$\frac{X(z)}{z} = \frac{(1-e^{-aT})}{(z-1)(z-e^{-aT})} = \frac{A}{(z-1)} + \frac{B}{(z-e^{-aT})}$$

$$\frac{(1-e^{-aT})}{\cancel{(z-1)(z-e^{-aT})}} = \frac{A(z-e^{-aT})+B(z-1)}{\cancel{(z-1)(z-e^{-aT})}}$$

$$1-e^{-aT} = Az - Ae^{-aT} + Bz - B$$

$$1-e^{-aT} = z(A+B) - Ae^{-aT} - B$$

$$A+B = 0 \longrightarrow (1)$$
$$-Ae^{-aT} - B = 1-e^{-aT} \longrightarrow (2)$$

$$A+\cancel{B} = 0$$
$$-Ae^{-aT} - \cancel{B} = 1-e^{-aT}$$
$$\overline{A - Ae^{-aT} = 1-e^{-aT}}$$

$$A(1-e^{-aT}) = (1-e^{-aT})$$

$$A = 1$$

$$A+B=0$$
$$1+B=0$$
$$B=-1$$

49. $\frac{X(z)}{z} = \frac{1}{(z-1)} + \frac{1}{(z-e^{-aT})}$

$$X(z) = \frac{z}{z-1} + \frac{z}{z-e^{-aT}}$$

By Applying Inverse z-T/F $\therefore\ z^{-1}\left[\frac{z}{z-1}\right] = 1$

$X(KT)$ $\quad z^{-1}\left[\frac{z}{z-e^{-aT}}\right] = e^{-aKT}$

$$\boxed{X(KT) = 1 - e^{-aKT}}$$

$K = 0, 1, 2$

$X(0) = 1 - e^{-0} = 1 - 1 = 0$

$X(1) = 1 - e^{-a(1)T} = 1 - e^{-aT}$, etc.

Pb4. $X(z) = \frac{z(z-2)}{(z+0.2)(z+0.6)}$ Calculate Inverse z-Transformatio

$$X(z) = \frac{z(z-2)}{(z+0.2)(z+0.6)}$$

$$\frac{X(z)}{z} = \frac{(z-2)}{(z+0.2)(z+0.6)} = \frac{A}{(z+0.2)} + \frac{B}{(z+0.6)}$$

$$\frac{z-2}{\cancel{(z+0.2)(z+0.6)}} = \frac{A(z+0.6) + B(z+0.2)}{\cancel{(z+0.2)(z+0.6)}}$$

600

400

1400

500

500

$$z-2 = A(z+0.6) + B(z+0.2)$$

$$z-2 = (A+B)z + 0.6A + 0.2B$$

$A + B = 1 \longrightarrow (1)$

$0.6A + 0.2B = -2 \longrightarrow (2)$

$$\frac{X(z)}{z} = \frac{\overset{-5.5}{\cancel{2.75}}}{(z+0.2)} + \frac{\overset{6.5}{\cancel{(-1.75)}}}{(z+0.6)}$$

$A + B = 1$
$0.6A + 0.2B = -2$ } By solving

$A = \cancel{2.75}\ -5.5$

$B = \cancel{-1.75}\ 6.5$

$$X(z) = \frac{z(\overset{-5.5}{\cancel{2.75}})}{z+0.2} + \frac{\overset{6.5}{\cancel{1.75}}\,z}{z+0.6}$$

By applying Inverse z-T/F

$$\boxed{X(KT) = -5.5(-0.2)^K + 6.5(-0.6)^K}$$

Pb5. $X(z) = \frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$ find Inverse Z-Transformation

$$\frac{X(z)}{z} = \frac{1-e^{-aT}}{(z-1)(z-e^{-aT})} = \frac{A}{(z-1)} + \frac{B}{(z-e^{-aT})}$$

$$\frac{1-e^{-aT}}{(z-1)(z-e^{-aT})} = \frac{A(z-e^{-aT})+B(z-1)}{(z-1)(z-e^{-aT})}$$

$$1-e^{-aT} = A(z-e^{-aT})+B(z-1)$$

$$A+B=0 \rightarrow ①$$

$$-Ae^{-aT}-B = 1-e^{-aT} \rightarrow ②$$

By solving ① & ②

$$A+\cancel{B}=0$$
$$-Ae^{-aT}-\cancel{B} = 1-e^{-aT}$$
$$A-Ae^{-aT} = 1-e^{-aT}$$
$$A(1-e^{-aT}) = 1-e$$

Pb5. Calculate Inverse Z-Transformation for the function $X(S) = \frac{1}{S(S+1)}$

Sol

$$X(S) = \frac{1}{S(S+1)}$$

$$X(S) = \frac{1}{S(S+1)} = \frac{A}{S} + \frac{B}{(S+1)}$$

$$\frac{1}{\cancel{S(S+1)}} = \frac{A(S+1)+BS}{\cancel{S(S+1)}}$$

$$AS+A+BS = 1$$

$$A+B=0$$

$$A=1$$

$$1+B=0$$

$$B=-1$$

$$X(S) = \frac{1}{S} - \frac{1}{S+1}$$

Apply Inverse Laplace Transform

$$x(t) = 1 - e^{-t}$$

put $t = KT$

$$\rightarrow \boxed{x(KT) = 1 - e^{-KT}}$$

z-Transform $\boxed{X(z) = \sum_{K=0}^{\infty} x(KT) z^{-K}}$

$$X(z) = \sum_{K=0}^{\infty} (1 - e^{-KT}) z^{-K}$$

$$= \sum_{K=0}^{\infty} 1 \cdot z^{-K} - \sum_{K=0}^{\infty} e^{-KT} \cdot z^{-K}$$

$$X(z) = \left(\frac{z}{z-1}\right) - \left(\frac{z}{z-e^{-T}}\right)$$

$$X(z) = \frac{z(z-e^{-T}) - z(z-1)}{(z-1)(z-e^{-T})}$$

* Inverse Integral Method:

This is a useful technique for obtaining z-transform. The 'inverse 'integral for z-Transformation $X(z)$ is

$$z^{-1}[X(z)] = x(KT) = \frac{1}{2\pi j} \oint X(z) \cdot z^{K-1} dz.$$

$$x(KT) = x(K) = K_1 + K_2 + K_3 + \cdots + K_m$$

$$= \sum_{i=1}^{m} \left[\text{residue of } X(z)\, z^{K-1} \text{ at pole } z = z_i \text{ of } X(z) \cdot z^{K-1}\right].$$

Where $K_1, K_2 \ldots K_m$ denote residue of $X(z) z^{K-1}$ at poles $z_1, z_2 \ldots z_m$ respectively.

Pbl. $\left[X(z) = \frac{10z}{(z-1)(z-2)}\right.$ Obtain Inverse z-Transform Using Inverse Integral Method.

Sol

$$x(KT) = \frac{1}{2\pi j} \oint X(z) \cdot z^{K-1} dz$$

$$= \frac{1}{2\pi j} \oint \left[\frac{10z}{(z-1)(z-2)}\right] z^{K-1} dz$$

$$= \frac{1}{2\pi j} \oint \left[\frac{A}{z-1} + \frac{B}{z-2}\right] z^{K-1} dz.$$

$A = -10, \ B = 10$

$$x(KT) = \frac{1}{2\pi j} \oint \left[\frac{-10z^K}{z-1} + \frac{10z^K}{z-2} \right] dz$$

$$= \left(\text{residue of } \frac{-10z^K}{z-1} \text{ at pole } z=1 \right)$$

$$+ \left(\text{residue of } \frac{10z^K}{z-2} \text{ at pole } z=2 \right)$$

$$x(KT) = 10\,(-1+2^K)$$

Pb2)

$$X(z) = \frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$$

$$X(z)\,z^{K-1} = \frac{(1-e^{-aT})z^K}{(z-1)(z-e^{-aT})}$$

for $K = 0, 1, 2 \cdots$ $X(z) \cdot z^{K-1}$ has 2 simple poles, $z = z_1 = 1$ & $z = z_2 = e^{-aT}$

Hence,

$$x(K) = \sum_{i=1}^{2} \left[\text{residue of } \frac{(1-e^{-aT})z^K}{(z-1)(z-e^{-aT})} \text{ at pole } z = z_i \right]$$

$$= K_1 + K_2$$

$$K_1 = \lim_{z \to 1} \left[(z-1) \frac{(1-e^{-aT})z^K}{(z-1)(z-e^{-aT})} \right] = 1$$

(Pole at $z=1$)

$$K_2 = \lim_{z \to e^{-aT}} \left[(z-e^{-aT}) \frac{(1-e^{-aT})z^K}{(z-1)(z-e^{-aT})} \right] = -e^{-KaT}$$

$$x(KT) = K_1 + K_2$$

$$\boxed{x(KT) = 1 - e^{-aKT}}$$

$K = 0, 1, 2 \cdots$

* Differential Equation Method: [Z-Transform Method]

$$Z[x(k)] = X(Z)$$

$$Z[x(k-1)] = Z^{-1}X(Z)$$

$$Z[x(k-2)] = Z^{-2}X(Z)$$

By using shifting Theorem.

$$Z[x(k-3)] = Z^{-3}X(Z)$$

$$Z[x(k+1)] = ZX(Z) - Zx(0)$$

$$Z[x(k+2)] = Z^2X(Z) - Z^2x(0) - Zx(1)$$

$$Z[x(k+3)] = Z^3X(Z) - Z^3x(0) - Z^2x(1) - Zx(2)$$

Pb1. solve the following difference Equation by use of the Z-Transform Method.

$$x(k+2) + 3x(k+1) + 2x(k) = 0, \quad x(0) = 0, \; x(1) = 1$$

Sol

$$x(k+2) + 3x(k+1) + 2x(k) = 0$$

$$Z[x(k+2)] = Z^2X(Z) - Z^2x(0) - Zx(1)$$

$$Z[x(k+1)] = ZX(Z) - Zx(0)$$

$$Z[x(k)] = X(Z)$$

$$\left[Z^2X(Z) - Z^2\cancel{x(0)}^{0} - Z\cancel{x(1)}^{1}\right] + 3\left[ZX(Z) - Z\cancel{x(0)}^{0}\right] + 2X(Z) = 0$$

$$Z^2X(Z) + 3ZX(Z) + 2X(Z) = 0 + Z$$

$$\left[Z^2 + 3Z + 2\right]X(Z) = Z$$

$$X(Z) = \frac{Z}{Z^2 + 3Z + 2} = \frac{Z}{(Z+1)(Z+2)}$$

$$\frac{X(Z)}{Z} = \frac{1}{(Z+1)(Z+2)} = \frac{A}{(Z+1)} + \frac{B}{(Z+2)}$$

$$\frac{1}{(Z+1)(Z+2)} = \frac{A}{(Z+1)} + \frac{B}{(Z+2)}$$

$$\frac{1}{\cancel{(Z+1)(Z+2)}} = \frac{A(Z+2) + B(Z+1)}{\cancel{(Z+1)(Z+2)}}$$

$$1 = A(Z+2) + B(Z+1)$$

$$1 = (A+B)Z + 2A + B$$

$$A + B = 0 \rightarrow (1)$$

$$2A + B = 1 \rightarrow (2)$$

$$-A = -1$$

$$A = 1; \quad B = -1$$

$$\frac{X(z)}{z} = \frac{1}{(z+1)} - \frac{1}{(z+2)}$$

$$X(z) = \frac{z}{z+1} - \frac{z}{z+2}$$

By applying Inverse z-Transformation.

$$\boxed{x(KT) = (-1)^K - (-2)^K}$$

Pb2. $x(K+2) - 3x(K+1) + 2x(K) = u(K)$ find the response $x(k)$ of the following system $x(K) = 0$ for $K \le 0$,

$u(0) = 1$

$u(K) = 0$ for $K<0$, $K>0$

Sol

$$x(K+2) - 3x(K+1) + 2x(K) = u(K)$$

$K = -1$

$$x(1) - 3x(0) + 2x(-1) = u(-1)$$

$$x(1) - 0 + 2(0) = 0$$

$$\boxed{x(1) = 0}$$

$$\left[z^2X(z) - \cancel{z^2x(0)} - \cancel{zx(1)}\right] + 3\left[zX(z) - \cancel{zx(0)}\right] + 2X(z) = u(z)$$

$$z^2X(z) - 3zX(z) + 2X(z) = u(z)$$

$$X(z) = \frac{u(z)}{z^2-3z+2} \qquad u(z) = 1$$

$$X(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$X(z) = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$z[x(K+1)] = zX(z) - \cancel{zx(0)}$$

$$= z\left[\frac{-1}{z-1} + \frac{1}{z-2}\right]$$

$$z[x(K+1)] = \frac{-z}{z-1} + \frac{z}{z-2}$$

$$x(K+1) = -1 + z^{K}, \quad K = 0,1,2\ldots$$

$$x(k) = -1 + z^{K-1}, \quad K = 0,1,2\ldots$$

Pb3. $x(K+2) - x(K+1) + 0.25\,x(K) = u(K+2)$

*PULSE TRANSFER FUNCTION :

Transfer-function relates laplace transform of continuous time output to that of the continuous time input

$$G(s) = \frac{C(s)}{R(s)}$$

pulse Transfer function relates the Z-Transform of the o/p to that of the Input

$$G(Z) = \frac{C(Z)}{R(Z)} \quad \text{(or)}$$

$$X(Z) \longrightarrow \boxed{G(Z)} \longrightarrow Y(Z)$$

$$G(Z) = \frac{Y(Z)}{X(Z)}$$ [pulse Transfer Function (or) Z-Transfer Function]

*General procedure For Obtaining pulse Transfer Function

1. Obtain the transfer Function $G(s)$ of the system
2. Obtain the impulse response function $g(t) = \mathcal{L}^{-1}[G(s)]$
3. Evaluate,

$$G(Z) = \sum_{K=0}^{\infty} g(KT)\, Z^{-K}.$$

Pb1. Obtain the pulse transfer Function $G(Z)$, if $G(s) = \frac{1}{s+a}$

sol

$$G(s) = \frac{1}{s+a}$$

Laplace Transform of $G(s) = e^{-at}$

let $KT = t$

$$G(KT) = e^{-aKT}.$$

then $G(Z) = \sum_{K=0}^{\infty} g(KT)\, Z^{-K}$

$$= \sum_{K=0}^{\infty} e^{-aKT} \cdot Z^{-K}$$

$$= \sum_{k=0}^{\infty} (e^{-aT}z^{-1})^{k}$$

$$G(z) = \frac{1}{1-e^{-aT}z^{-1}}$$

2. $G(s) = \frac{K}{(s+a)(s+b)}$ find $G(z)$

$$G(s) = \frac{K}{(s+a)(s+b)} = \frac{A}{(s+a)} + \frac{B}{(s+b)}$$

$$\frac{K}{(s+a)\cancel{(s+b)}} = \frac{A(s+b)+B(s+a)}{\cancel{(s+a)}(s+b)}$$

$$K = A(s+b)+B(s+a)$$

$$K = (A+B)s + Ab + Ba$$

$$A+B=0 \rightarrow (1)$$

$$Ab+Ba=k \rightarrow (2)$$

By solving (1) & (2)

$$A\cancel{b}+Bb=0$$

$$\underline{\cancel{A}b+Ba=K}$$

$$B(b-a) = -K$$

$$B = -K/b-a = \frac{K}{a-b}$$

$$A+B=0$$

$$A+\frac{K}{a-b}=0$$

$$A = -\frac{K}{a-b} = \frac{K}{b-a}$$

$$G(s) = \frac{(K/b-a)}{s+a} + \frac{\left(\frac{K}{a-b}\right)}{s+b}$$

$$G(s) = \frac{K}{(b-a)}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$g(t) = L^{-1}[G(s)] = L^{-1}\left[\left(\frac{K}{b-a}\right)\left(\frac{1}{s+a} - \frac{1}{s+b}\right)\right]$$

$$g(t) = \frac{K}{b-a}\left[e^{-at} - e^{-bt}\right]$$

$$g(KT) = \frac{K}{b-a}\left[e^{-aKT} - e^{-bKT}\right]$$

$$G(z) = \sum_{K=0}^{\infty} g(KT) z^{-K}$$

$$= \sum_{K=0}^{\infty} \frac{K}{b-a}\left[e^{-aKT} - e^{-bKT}\right] z^{-K}$$

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$$G(z) = \frac{k}{b-a}\left[\sum_{k=0}^{\infty} e^{-akT} z^{-k} - \sum_{k=0}^{\infty} e^{-bkT} z^{-k}\right]$$

$$= \frac{k}{b-a}\left[\frac{1}{1-e^{-aT}z^{-1}} - \frac{1}{1-e^{-bT}z^{-1}}\right]$$

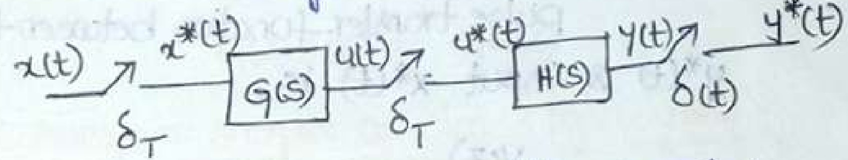
$$= \frac{k}{b-a}\left[\frac{\cancel{1}-e^{-bT}z^{-1} - \cancel{1}+e^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}\right]$$

$$G(z) = \frac{k}{b-a}\left[\frac{e^{-aT}z^{-1} - e^{-bT}z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}\right]$$

3. Obtain the [Z-Transform analysis] pulse Transfer Function of $G(s) = \frac{1-e^{-s}}{s^2(s+1)}$

* Z-Transform Analysis of open Loop systems:

(i) Consider a system shown below



Sampled system with a sampler

pulse transfer function of the open loop system shown above is $G(z)H(z)$

From the diagram

$$U(s) = G(s)\,X^*(s) \longrightarrow ①$$
$$Y(s) = H(s)\,U^*(s) \longrightarrow ②$$

By taking starred Laplace Transform of each of these 2 equation, we get

$$U^*(s) = G^*(s)\,X^*(s)$$
$$Y^*(s) = H^*(s)U^*(s) = H^*(s)\left[G^*(s)X^*(s)\right]$$

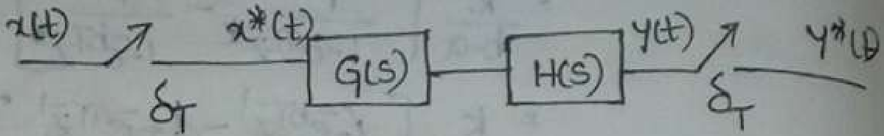
Interms of Z-Transform

$$Y(z) = H(z)\,G(z)\,X(z)$$

pulse transfer function

$$\frac{Y(z)}{X(z)} = H(z)\,G(z)$$

(ii) Consider open Loop system, without a sampler



From the diagram

$$Y(S) = G(S)H(S)X^*(S) \longrightarrow ①$$

$$Y(S) = GH(S)\,X^*(S) \longrightarrow ②$$

where $GH(S) = G(S)\,H(S)$

Taking the starred Laplace transform of $Y(S)$, ②

$$Y^*(S) = [GH(S)]^*\,X^*(S)$$

"interms of Z-Transform Notation

$$Y(Z) = GH(Z)\,X(Z)$$

pulse transfer function between the output $y^*(t)$ & input $x^*(t)$ is,

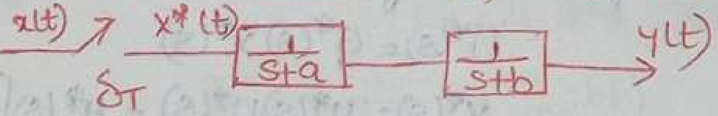
$$\frac{Y(Z)}{X(Z)} = GH(Z)$$

$$= Z[GH(S)]$$

$$G(Z)\,H(Z) \neq GH(Z)$$

$$Z[G(S)H(S)] \neq Z[GH(S)]$$

Pb1) Obtain the pulse transfer function $Y(Z)/X(Z)$ for each of these systems



a) For this system,

$$\frac{Y(Z)}{X(Z)} = \frac{Y(Z)}{U(Z)}\cdot\frac{U(Z)}{X(Z)} = H(Z)\cdot G(Z).$$

$$G(Z) = Z\left[\frac{1}{S+a}\right] = \frac{1}{1-e^{-aT}Z^{-1}}$$

$$H(Z) = Z\left[\frac{1}{S+b}\right] = \frac{1}{1-e^{-bT}Z^{-1}}$$

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$$\frac{Y(z)}{X(z)} = G(z)H(z) = \frac{1}{1-e^{-aT}z^{-1}} \cdot \frac{1}{1-e^{-bT}z^{-1}} \rightarrow (1)$$

b) This system is not separated by a sampler

$$\frac{Y(z)}{X(z)} = GH(z)$$

$$= Z[G(s)H(s)]$$

$$= Z\left[\frac{1}{s+a} \cdot \frac{1}{s+b}\right] \qquad \frac{A}{s+a} + \frac{B}{s+b}$$

$$= Z\left\{\frac{1}{b-a}\left(\frac{1}{s+a} - \frac{1}{s+b}\right)\right\} \qquad A = \frac{1}{b-a}$$

$$= \frac{1}{b-a}\left(\frac{1}{1-e^{-aT}z^{-1}} - \frac{1}{1-e^{-bT}z^{-1}}\right) \qquad B = -\frac{1}{b-a}$$

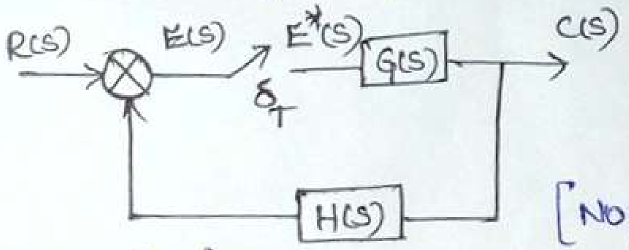
$$GH(z) = \frac{(e^{-aT} - e^{-bT})z^{-1}}{(b-a)(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})} \rightarrow (2)$$

$$(1) \neq (2)$$

$$G(z)H(z) \neq GH(z)$$

* Z-Transform Analysis of closed loop system

In a closed loop system the existence (or) Non existance of an o/p sampler within the loop makes a difference in the behaviour of the system



[No sampler b/w G(s) & H(s)]

$$E(s) = R(s) - H(s)C(s)$$

$$C(s) = G(s)E^*(s)$$

Hence, $E(s) = R(s) - H(s)G(s)E^*(s)$

By taking starred Laplace transform

$$E^*(s) = R^*(s) - H^*(s)G^*(s)E^*(s)$$

$$= R^*(s) - GH^*(s)E^*(s)$$

$$E^*(s) = \frac{R^*(s)}{1+GH^*(s)}$$

$$C^*(s) = G^*(s)\,E^*(s)$$

$$C^*(s) = \frac{G^*(s)R^*(s)}{1+GH^*(s)}$$

Interms of Z-Transform Notation,

$$C(z) = \frac{G(z)R(z)}{1+GH(z)}.$$

pulse Transfer Function $\boxed{\dfrac{C(z)}{R(z)} = \dfrac{G(z)}{1+GH(z)}}$

UNIT-III

STATE SPACE ANALYSIS AND THE CONCEPTS OF CONTROLLABILITY & OBSERVABILITY

For the Analysis & design of control system there are 2 types of Methods

1. Conventional Methods
2. State space Analysis.

1. Conventional Methods such as Root locus, frequency Response methods are useful for dealing with single i/p, single o/p systems.
2. These are Applicable for Linear, time invarient systems
3. Initial conditions are not taken.

***Advantages of state space Analysis:**

1. A modern control system may have many i/ps and many outputs and these ~~to~~ may be interrelated in a Complicated manner. the state space methods for the analysis & synthesis of control systems are best suited for those systems
2. Natural and convenient for computer solution
3. Applicable for Non Linear & time varient systems.
4. Initial conditions are also considered.

*** Concept of state space Method:**

State: The state of the dynamic system is the smallest set of variables (state variables) such that knowledge of these variables at $t = t_0$, together with the knowledge of input for $t \geq t_0$, completely determines the behaviour of the system at any time $t \geq t_0$.

State Variable: The state variables of a dynamic system are the variables that determines the state of a dynamic system.

Atleast n variables $x_1, x_2 \cdots x_n$ are needed to completely describe the behaviour of a dynamic system.

State vector: The 'n' state variables can be considered the 'n' components of a vector. Such vector is called as a state vector.

State space: The n-dimensional space whose coordinate axes consists of the x_1 axis, x_2 axis $\cdots x_n$ is called a space of states or state space.

State space Equations: In state space, there are 3 main variables input variables, o/p variables & state variables.

State Equation $x(k+1) = f[x(k), u(k), k]$

o/p Equation $y(k) = g[x(k), u(k), k]$

The combination of above 2 equations is called as "State Model".

$$x(k+1) = G(k)x(k) + H(k)u(k)$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

Where $x(k) = n$ state vector
$y(k)$ = output vector, m
$u(k)$ = input vector, r
$G(k)$ = state Matrix, $n \times n$
$H(k)$ = input Matrix, $n \times r$
$C(k)$ = output Matrix, $m \times n$
$D(k)$ = direct transmission Matrix, $m \times r$

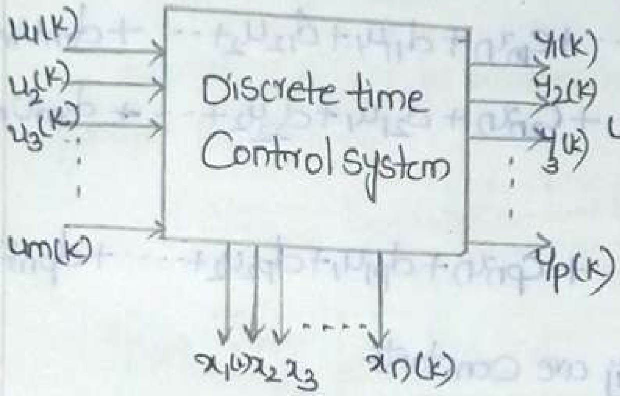
If the system is invarient

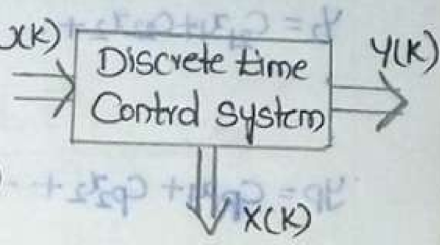
$$x(k+1) = Gx(k) + Hu(k)$$
$$y(k) = Cx(k) + Du(k)$$

State Space Representation of Discrete time Systems:

Consider a system Consisting of 'm'-inputs, 'p'-outputs & 'n'- state variables.

Let $x_1(K), x_2(K) ---- x_n(K)$ — state Variables
$u_1(K), u_2(K) ----- u_m(K)$ — input variables
$y_1(K), y_2(K) ----- y_p(K)$ — output Variables.





Where $u(K) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$; $x(K) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$; $y(K) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$

The state equation of a discrete time system is a set of n-Number first order difference equations

$$x_1(1+K) = f_1[x_1, x_2, x_3 -- x_n ; u_1, u_2, u_3 -- u_m]$$
$$x_2(1+K) = f_2[x_1, x_2, x_3 -- x_n ; u_1, u_2, u_3 --- u_m]$$
$$\vdots$$
$$x_n(1+K) = f_n[x_1, x_2, x_3 --- x_n ; u_1, u_2, u_3 --- u_m]$$

$$x_1(1+K) = a_{11}x_1 + a_{12}x_2 + --- + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + --- + b_{1m}u_m$$
$$x_2(1+K) = a_{21}x_1 + a_{22}x_2 + --- + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + --- + b_{2m}u_m$$
$$\vdots$$
$$x_n(1+K) = a_{n1}x_1 + a_{n2}x_2 + --- + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + --- + b_{nm}u_m$$

Where the coefficients a_{ij} & b_{ij} are constants

$$\begin{bmatrix} x_1(K+1) \\ x_2(K+1) \\ \vdots \\ x_n(K+1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & ---- & a_{1n} \\ \vdots \\ \vdots \\ a_{n1} & a_{n2} & ---- & a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} + \begin{bmatrix} b_{11} & b_{12} & -- & b_{1m} \\ \vdots \\ \vdots \\ b_{n1} & b_{n2} & -- & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$\boxed{X(K+1) = Ax(K) + BU(K)}$$

$X(K)$ = state vector of or $n\times1$

$U(K)$ = Input vector, $m\times1$

A = system Matrix of order $n\times n$

B = Input Matrix, $n\times m$

The output of any discrete time instant, K

$$y_1 = c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \cdots + d_{1m}u_m$$

$$y_2 = c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \cdots + d_{2m}u_m$$

$$\vdots$$

$$y_p = c_{p1}x_1 + c_{p2}x_2 + \cdots + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \cdots + d_{pm}u_m$$

where c_{ij} & d_{ij} are constants.

$$\underset{p\times1}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}} = \underset{p\times n}{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix}} \underset{n\times1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} + \underset{p\times m}{\begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & & & \\ d_{p1} & d_{p2} & \cdots & d_{pn} \end{bmatrix}} \underset{m\times1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}}$$

$$\boxed{Y(K) = CX(K) + DU(K)}$$

where $Y(K)$ = o/p vector, $p\times1$

C = o/p Matrix, $p\times n$

D = transmission matrix, $p\times m$

The state equation & o/p equation together called as state model of the system

$$X(K+1) = AX(K) + BU(K) \quad \text{(State Equation)}$$

$$Y(K) = CX(K) + DU(K) \quad \text{(o/p Equation)}$$

67 * Various Representations

* Canonical Forms for discrete time state space Equation:

Consider the discrete time system described by

$$y(k) = a_1 y(k-1) + a_2 y(k-2) + \cdots + a_n y(k-n) + b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n) \rightarrow ①$$

Where $u(k)$ is the input & $y(k)$ is the o/p of the system at kth sampling instant.

Equation ① can be written in the form of the pulse transfer function as

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} \rightarrow ②$$

$$\text{[or]} \quad \frac{Y(z)}{U(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n} \rightarrow ③$$

1. Controllable Canonical Form:

The state space representation of the discrete time system may be put in the form given by following equation

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \left[b_n - a_n b_0 \,\vdots\, b_{n-1} - a_{n-1} b_0 \,\vdots\, \cdots \,\vdots\, b_1 - a_1 b_0 \right] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

2. Observable Canonical Form:

The state space Representation of the discrete time system given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 & 0 & \cdots & -a_{n-1} \\ \vdots & & & & & & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & -a_2 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

3. **Diagonal Canonical Form:**

If the poles of the pulse transfer function given

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & \cdots & p_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

4) **Jordan Canonical Form:** If the pulse transfer function given by equations ①, ② & ③ involves a multiple pole of order 'm' at $z = p_1$ & all other poles are distinct, then the state equation & o/p equation may be given as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_m(k+1) \\ x_{m+1}(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \left[\begin{array}{ccccc|ccc} p_1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & p_1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & \cdots & p_1 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & p_{m+1} & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & p_n \end{array}\right] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_m(k) \\ x_{m+1}(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \hline 1 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + b_0 u(k)$$

69 1. Obtain a state space representation of the system described by $y(k+2)+y(k+1)+0.16y(k) = u(k+1)+2u(k)$

Sol: Define state variables

$$x_1(k) = y(k) \; ; \; x_2(k) = x_1(k+1) - u(k)$$

$$x_1(k+1) = x_2(k) + u(k)$$

$$x_2(k+1) = -0.16x_1(k) - x_2(k) + u(k)$$

$$y(k) = x_1(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

2. Consider the following system

$$\frac{Y(z)}{U(z)} = \frac{z+1}{z^2+1.3z+0.4}$$

Sol ① Controllable canonical form :

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \; ; \quad y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

② observable canonical form :

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -0.4 \\ 1 & -1.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

③ Diagonal Canonical Form :

$$\frac{Y(z)}{U(z)} = \frac{5/3}{(z+0.5)} + \frac{-2/3}{(z+0.8)} \qquad \frac{z+1}{z^2+1.3z+0.4} = \frac{A}{(z+0.5)} + \frac{B}{(z+0.8)}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.8 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 5/3 & -2/3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

PULSE TRANSFER FUNCTION MATRIX:

A single I/p single o/p discrete system may be modeled by a "pulse Transfer Function".

A Multi I/p Multi o/p discrete system gives "The pulse transfer Function Matrix".

Consider State Model

$$x(k+1) = Gx(k) + Hu(k) \longrightarrow (1)$$
$$y(k) = Cx(k) + Du(k) \longrightarrow (2)$$

By Taking Z-Transformation

$$zX(z) - zx(0) = GX(z) + HU(z) \longrightarrow (3)$$
$$Y(z) = CX(z) + DU(z) \longrightarrow (4)$$

Assume zero initial conditions $x(0)=0$; then from (3) eq

$$zX(z) - 0 = GX(z) + HU(z)$$
$$zX(z) - GX(z) = HU(z)$$
$$X(z)[zI-G] = HU(z)$$
$$X(z) = [zI-G]^{-1}HU(z) \longrightarrow (5)$$

From equation (4) & (5)

$$Y(z) = C\left[[zI-G]^{-1}HU(z)\right] + DU(z)$$
$$Y(z) = \left[C[zI-G]^{-1}H + D\right]U(z)$$
$$\frac{Y(z)}{U(z)} = C[zI-G]^{-1}H + D$$

So Pulse Transfer Function $\boxed{\frac{Y(z)}{U(z)} = C[zI-G]^{-1}H+D}$

Where $[zI-G]^{-1} = \frac{Adj(zI-G)}{|zI-G|}$

Characteristic Equation $|zI-G| = 0$

71 * STATE TRANSITION MATRIX :(2)

Consider the discrete-time system

$$x(K+1) = Gx(K) + HU(K)$$

By taking the z-Transform on both sides

$$zX(z) - zX(0) = GX(z) + HU(z)$$

$$zX(z) - GX(z) = zX(0) + HU(z)$$

$$X(z)\left[zI - G\right] = zX(0) + HU(z)$$

premultiplying on both sides by $(zI-G^{-1})$ for above equation

$$[zI-G]^{-1} X(z)\,[zI-G] = [zI-G]^{-1} z\,X(0) + [zI-G]^{-1} HU(z)$$

$$X(z) = [zI-G]^{-1} z\,X(0) + [zI-G]^{-1} HU(z)$$

By Taking Inverse z-Transform of both sides for above equation

$$X(K) = z^{-1}\left[[zI-G]^{-1} z\right] X(0) + z^{-1}\left[[zI-G]^{-1} HU(z)\right]$$

where $\boxed{G^K = z^{-1}\left[(zI-G)^{-1} z\right]}$

state Transition Matrix $\boxed{\phi(K) = G^K = z^{-1}\left[(zI-G)^{-1} z\right]}$

* State-transition Matrix properties :—

The state transition Matrix of the system

$$x(K+1) = Gx(K) + HU(K)$$

$$\phi(K) = G^K$$

1. $\phi(0) = I$

$$\phi(K) = G^K \Rightarrow \phi(0) = G^0 = 1 = I$$

2. $\phi^{-1}(K) = \phi(-K)$

$$\phi(K) = G(K) = G^K$$

$$\phi^{-1}(K) = (G^K)^{-1} = G^{-K} = \phi(-K)$$

$$\therefore \phi^{-1}(K) = \phi(-K)$$

3. $\phi(K_1, K_0) = \phi(K-K_0) = G^{K-K_0}$ $\quad \phi(K) = G^K,\ \phi(K-K_0) = G^{K-K_0}$

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4. $\phi(K_1-K_2)\,\phi(K_2-K_3) = \phi(K_1-K_3)$

$$\phi(K_1-K_2) = G^{K_1-K_2} \;;\; \phi(K_2-K_3) = G^{K_2-K_3}$$

$$\phi(K_1-K_2)\,\phi(K_2-K_3) = G^{K_1-K_2}\cdot G^{K_2-K_3} = G^{K_1-K_3}$$

$$\frac{G^{K_1}}{\cancel{G^{K_2}}}\cdot\frac{\cancel{G^{K_2}}}{G^{K_3}}$$

5. $\phi(K_1+K_2) = \phi(K_1)\cdot\phi(K_2)$

$$\phi(K) = G^K, \quad \phi(K_1+K_2) = G^{K_1+K_2} = G^{K_1}\cdot G^{K_2}$$

$$= \phi(K_1)\cdot\phi(K_2).$$

1) Obtain state transition matrix of the following discrete time system $x(K+1) = Gx(K) + Hu(K)$

where $G = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix}$, $H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Sol: State transition Matrix

$$\boxed{\phi(K) = G^K = Z^{-1}\left[(ZI-G)^{-1}Z\right]}$$

$$[ZI-G]^{-1} = Z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} = \begin{bmatrix} Z & -1 \\ 0.16 & Z+1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} Z & -1 \\ 0.16 & Z+1 \end{bmatrix}^{-1} = \frac{1}{Z(Z+1)+1(0.16)}\begin{bmatrix} Z+1 & 1 \\ -0.16 & Z \end{bmatrix} = \begin{bmatrix} \frac{Z+1}{(Z+0.2)(Z+0.8)} & \frac{1}{(Z+0.2)(Z+0.8)} \\ \frac{-0.16}{(Z+0.2)(Z+0.8)} & \frac{Z}{(Z+0.2)(Z+0.8)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{A}{(Z+0.2)} + \frac{B}{(Z+0.8)} & \frac{A}{(Z+0.2)} + \frac{B}{(Z+0.8)} \\ \frac{A}{(Z+0.2)} + \frac{B}{(Z+0.8)} & \frac{A}{(Z+0.2)} + \frac{B}{(Z+0.8)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4/3}{(Z+0.2)} + \frac{-1/3}{(Z+0.8)} & \frac{5/3}{(Z+0.2)} + \frac{-5/3}{(Z+0.8)} \\ \frac{-0.8/3}{Z+0.2} + \frac{0.8/3}{Z+0.8} & \frac{-1/3}{Z+0.2} + \frac{4/3}{Z+0.8} \end{bmatrix}$$

$$\#3 \quad (zI-G)^{-1}z = \begin{bmatrix} \frac{4/3\, z}{(z+0.2)} + \frac{-1/3\, z}{(z+0.8)} & \frac{5/3\, z}{(z+0.2)} + \frac{-5/3\, z}{(z+0.8)} \\ \frac{-0.8/3\, z}{(z+0.2)} + \frac{0.8/3\, z}{(z+0.8)} & \frac{-1/3\, z}{(z+0.2)} + \frac{4/3\, z}{(z+0.8)} \end{bmatrix}$$

$$z^{-1}\left[(zI-G)^{-1}z\right] = \begin{bmatrix} 4/3(-0.2)^k - 1/3(-0.8)^k & 5/3(-0.2)^k - 5/3(-0.8)^k \\ -\frac{0.8}{3}(-0.2)^k - \frac{0.8}{3}(-0.8)^k & -1/3(-0.2)^k - 4/3(-0.8)^k \end{bmatrix}$$

to compute $x(k)$

$$z[x(k)] = X(z) = [zI-G]^{-1} zx(0) + [zI-G]^{-1} HU(z)$$

$$z^{-1}\left[\frac{z}{z-0.2}\right] = (0.2)^k$$

$$z^{-1}\left[\frac{z}{z+0.2}\right] = (-0.2)^k$$

$$U(z) = \frac{z}{z-1}$$

$$zx(0) + HU(z) = \begin{bmatrix} z \\ -z \end{bmatrix} + \begin{bmatrix} z/z-1 \\ z/z-1 \end{bmatrix} = \begin{bmatrix} z^2/z-1 \\ \frac{-z^2+2z}{z-1} \end{bmatrix}$$

Hence $X(z) = [zI-G]^{-1}\left[zx(0) + HU(z)\right]$

$$X(z) = \begin{bmatrix} \frac{z+1}{(z+0.2)(z+0.8)} & \frac{1}{(z+0.2)(z+0.8)} \\ \frac{-0.16}{(z+0.2)(z+0.8)} & \frac{z}{(z+0.2)(z+0.8)} \end{bmatrix} \begin{bmatrix} \frac{z^2}{(z-1)} \\ \frac{-z^2+2z}{z-1} \end{bmatrix}$$

$$X(z) = \begin{bmatrix} \frac{(z+2)z}{(z+0.2)(z+0.8)(z-1)} \\ \frac{(-z^2+1.84z)z}{(z+0.2)(z+0.8)(z-1)} \end{bmatrix} = \begin{bmatrix} \frac{A}{(z+0.2)} + \frac{B}{(z+0.8)} + \frac{C}{(z-1)} \\ \frac{D}{(z+0.2)} + \frac{E}{(z+0.8)} + \frac{F}{(z-1)} \end{bmatrix}$$

$$X(z) = \begin{bmatrix} -17/6\, z/z+0.2 + \frac{22}{9}\, z/z+0.8 + \frac{25}{18}\, z/z-1 \\ 3.4/6\, \frac{z}{z+0.2} - \frac{17.6}{9}\, z/z+0.8 + 7/18\, z/z-1 \end{bmatrix}$$

$$X(k) = z^{-1}[X(z)] = \begin{bmatrix} -17/6(-0.2)^k + 22/9(-0.8)^k + 25/18 \\ \frac{3.4}{6}(-0.2)^k - \frac{17.6}{9}(-0.8)^k + 7/18 \end{bmatrix}$$

* Methods for computation of State transition Matrix:

Methods of Evolution:

$(zI-G)^{-1}$ is easy in simple cases, generally a time consuming task.

$$[zI-G]^{-1} = \frac{Adj(zI-G)}{|zI-G|} \longrightarrow (1)$$

Note that the (determination |z) determinant $|zI-G|$

$$|zI-G| = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \longrightarrow (2)$$

$$Adj(zI-G) = Iz^{n-1} + H_1 z^{n-2} + H_2 z^{n-3} + \cdots + H_{n-1}$$

Where
$$\begin{cases} H_1 = G + a_1 I \\ H_2 = GH_1 + a_2 I \\ \vdots \\ H_{n-1} = GH_{n-2} + a_{n-1} I \\ H_n = GH_{n-1} + a_n I = 0 \end{cases}$$

$a_1, a_2 \cdots a_n$ are the coefficients appearing in the determinant

$$\begin{cases} a_1 = -trG \\ a_2 = -\frac{1}{2} tr GH_1 \\ a_3 = -\frac{1}{3} tr GH_2 \\ \vdots \\ a_n = -\frac{1}{n} tr GH_{n-1} \end{cases}$$

Pb) Determine the Inverse of the Matrix $(zI-G)$ where

$$G = \begin{bmatrix} 0.1 & 0.1 & 0 \\ 0.3 & -0.1 & -0.2 \\ 0 & 0 & -0.3 \end{bmatrix}$$ Also, obtain G^k

$n=3$

Sol

$$(zI-G)^{-1} = \frac{adj(zI-G)}{|zI-G|} = \frac{Iz^2 + H_1 z + H_2}{z^3 + a_1 z^2 + a_2 z + a_3}$$

1) $a_1 = -trG = -tr \begin{bmatrix} 0.1 & 0.1 & 0 \\ 0.3 & -0.1 & -0.2 \\ 0 & 0 & -0.3 \end{bmatrix} = 0.1 - 0.1 - 0.3$
(trace G)

$= -0.3$

$a_1 = -(-0.3) = 0.3$

75 2) $H_1 = G + a_1 I = \begin{bmatrix} 0.1 & 0.1 & 0 \\ 0.3 & -0.1 & -0.2 \\ 0 & 0 & -0.3 \end{bmatrix} + \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}$

a_1
H_1
a_2
H_2

$$H_1 = \begin{bmatrix} 0.4 & 0.1 & 0 \\ 0.3 & 0.2 & -0.2 \\ 0 & 0 & 0 \end{bmatrix}$$

3) $a_2 = -\frac{1}{2} tr\, GH_1 = -\frac{1}{2} tr \left\{ \begin{bmatrix} 0.1 & 0.1 & 0 \\ 0.3 & -0.1 & -0.2 \\ 0 & 0 & -0.3 \end{bmatrix} \begin{bmatrix} 0.4 & 0.1 & 0 \\ 0.3 & 0.2 & -0.2 \\ 0 & 0 & 0 \end{bmatrix} \right\}$

$$= -\frac{1}{2} tr \begin{bmatrix} 0.07 & 0.03 & -0.02 \\ 0.09 & 0.01 & 0.02 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= -0.04$$

4) $H_2 = GH_1 + a_2 I = \begin{bmatrix} 0.1 & 0.1 & 0 \\ 0.3 & -0.1 & -0.2 \\ 0 & 0 & -0.3 \end{bmatrix} \begin{bmatrix} 0.4 & 0.1 & 0 \\ 0.3 & 0.2 & -0.2 \\ 0 & 0 & 0 \end{bmatrix} + (-0.04) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0.03 & 0.03 & -0.02 \\ 0.09 & -0.03 & 0.02 \\ 0 & 0 & -0.04 \end{bmatrix}$$

5) $a_3 = -\frac{1}{3} tr\, GH_2 = -\frac{1}{3} tr \begin{bmatrix} 0.012 & 0 & 0 \\ 0 & 0.012 & 0 \\ 0 & 0 & 0.012 \end{bmatrix} = 0.012$

$$Adj\,|zI - G| = Iz^2 + H_1 z + H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} z^2 + \begin{bmatrix} 0.4 & 0.1 & 0 \\ 0.3 & 0.2 & -0.2 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0.03 & 0.03 & -0.02 \\ 0.09 & -0.03 & 0.02 \\ 0 & 0 & -0.04 \end{bmatrix}$$

$$Adj(ZI-G) = \begin{bmatrix} Z^2+0.4Z+0.03 & 0.1Z+0.03 & -0.02 \\ 0.3Z+0.09 & Z^2+0.2Z-0.03 & -0.2Z+0.02 \\ 0 & 0 & Z^2-0.04 \end{bmatrix} \rightarrow (1)$$

$$|ZI-G| = z^3+a_1z^2+a_2z+a_3 = z^3+0.3z^2+(-0.04)z-0.012$$

$$= (z+0.2)(z-0.2)(z+0.3) \rightarrow (2)$$

$$(ZI-G)^{-1} = \frac{Adj(ZI-G)}{|ZI-G|} = (1)/(2)$$

✱ Discretization of Continuous-time state space Equations:-

To Compute the state $x(t)$ by the use of digital computer, it must be converted into a discrete-time state equation

Consider the Continuous time state equation

$$\dot{x} = Ax+Bu \longrightarrow (1)$$

the discrete-time representation of the above equation

$$x(K+1)T = G(T)x(KT)+H(T)u(KT) \longrightarrow (2)$$

In order to determine $G(T)$ & $H(T)$, Consider the solution of the above equation (1), which is

$$x(t) = e^{At}x(0)+e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau \longrightarrow (3)$$

$$x(K+1)T = e^{A(K+1)T}x(0)+e^{A(K+1)T}\int_0^{(K+1)T} e^{-A\tau}Bu(\tau)d\tau \longrightarrow (4)$$

& where

$$x(KT) = e^{AKT}x(0)+e^{AKT}\int_0^{KT} e^{-A\tau}Bu(\tau)d\tau \longrightarrow (5)$$

Where x = state vector

u = I/p vector

$\dot{x}(t) = Ax(t)+Bu(t)$

$\dot{x}(t) - Ax(t) = Bu(t)$

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Multiply on both sides by e^{-At}

$$e^{-At}\left[\dot{x}(t) - Ax(t)\right] = e^{-At} \cdot BU(t)$$

$$\frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At} \cdot Bu(t)$$

Integrating on both sides b/w 0 & t

$$\int \frac{d}{dt}\left(e^{-At}x(t)\right) = \int_0^t e^{-A\tau} BU(\tau)d\tau$$

$$e^{-At}x(t) = x(0) + \int_0^t e^{-A\tau} BU(\tau)d\tau$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} BU(\tau)d\tau \longrightarrow (1)$$

put $t = (K+1)T$

$$x[(K+1)T] = e^{A(K+1)T}x(0) + \int_0^{(K+1)T} e^{A[(K+1)T-\tau]} BU(\tau)d\tau$$

$$= e^{A(K+1)T}x(0) + \int_0^{(K+1)T} e^{A(K+1)T} e^{-A\tau} BU(\tau)d\tau \longrightarrow (2)$$

put $t = KT$

$$x(KT) = e^{AKT}x(0) + \int_0^{KT} e^{A(KT-\tau)} BU(\tau)d\tau$$

$$x(KT) = e^{AKT}x(0) + e^{AKT}\int_0^{KT} e^{-A\tau} BU(\tau)d\tau \longrightarrow (3)$$

Multiply (3) equ with e^{AT}

$$x(KT)e^{AT} = e^{AKT} \cdot e^{AT} x(0) + e^{AKT} \cdot e^{AT}\int_0^{KT} e^{-A\tau} BU(\tau)d\tau$$

$$x(KT)e^{AT} = e^{A(K+1)T}x(0) + e^{A(K+1)T}\int_0^{KT} e^{-A\tau} BU(\tau)d\tau \longrightarrow (4)$$

(2)-(4) gives

$$\left[x(K+1)T - x(KT)e^{AT}\right] = \left\{\left(\cancel{e^{A(K+1)T}x(0)} + e^{A(K+1)T}\int_0^{(K+1)T} e^{-A\tau} BU(\tau)d\tau\right) - \left(\cancel{e^{A(K+1)T}x(0)} + e^{A(K+1)T}\int_0^{KT} e^{-A\tau} BU(\tau)d\tau\right)\right\}$$

$$x(K+1)T - x(KT)e^{AT} = e^{A(K+1)T}\int_{KT}^{(K+1)T} e^{-A\tau} BU(\tau)d\tau$$

$$x(K+1)T = x(KT)e^{AT} + e^{A(K+1)T}\int_{KT}^{(K+1)T} e^{-A\tau} BU(\tau)d\tau$$

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where $G(T) = e^{AT}$ ✓

$$H(T) = \int_0^T (e^{AT} d\lambda) B$$ ✓

$$x(k+1)T = G(T)x(kT) + H(T)\,U(kT)$$

→ Obtain a discrete-time state-space representation of the following continuous-time system & pulse transfer function

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Sol The desired discrete-time state equation is

$$x(k+1)T = G(T)x(kT) + H(T)u(kT)$$

$$G(T) = e^{AT} \; ; \; H(T) = \left(\int_0^T e^{AT} dt\right) B$$

$$e^{AT} = L^{-1}\left[SI - A\right]^{-1}$$

$$(SI-A)^{-1} = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} S & -1 \\ 0 & S+2 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} S & -1 \\ 0 & S+2 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} S+2 & 1 \\ 0 & S \end{bmatrix}}{S(S+2) - 0(-1)} = \begin{bmatrix} \frac{\cancel{S+2}}{S\cancel{(S+2)}} & \frac{1}{S(S+2)} \\ 0 & \frac{\cancel{S}}{\cancel{S}(S+2)} \end{bmatrix}$$

$$L^{-1}\left[SI - A\right]^{-1} = L^{-1}\begin{bmatrix} 1/S & 1/S(S+2) \\ 0 & 1/(S+2) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

$$G(T) = e^{AT} = \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2t}) \\ 0 & e^{-2T} \end{bmatrix}$$

$$H(T) = \left(\int_0^T e^{AT} dt\right) B = \int_0^T \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2T}) \\ 0 & e^{-2T} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt$$

$$= \int_0^T \begin{bmatrix} \frac{1}{2}(1-e^{-2T}) \\ e^{-2T} \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{1}{2}\left(T + e^{-2T/2}\right) \\ e^{-2T/2} \end{bmatrix}_0^T = \begin{bmatrix} \frac{1}{2}\left(T + \frac{e^{-2T}-1}{2}\right) \\ \frac{1}{2}\left(1 - e^{-2T}\right) \end{bmatrix}$$

Pulse Transfer Function :

$$PTF = C[SI-A]^{-1}B+D$$

Concept of Controllability & Observability :

* **Controllability** : A control system is said to be controllable if it is possible by means of an unbounded control vector to transfer the system from any initial state to any other state in a finite number of sampling periods.

It is the solution for pole placement problem

Consider the discrete time control system

$$x(K+1)T = Gx(KT) + HU(KT)$$

[∴ It is also called as State Controllability]

$X(KT)$ = state vector

$U(KT)$ = Control signal

$G = n \times n$ Matrix

$H = n \times 1$ Matrix [colomn Matrix]

T = sampling period

$$\Rightarrow \boxed{\text{Rank of } [H \,\vdots\, GH \,\vdots\, G^2H \,\vdots\, --- \,\vdots\, G^{n-1}H] = n}$$

where n = Rank of matrix "G"

For $G = 2 \times 2$ condition is $\text{Rank}[H \,\vdots\, GH] = 2$

For $G = 3 \times 3$ " " $\text{Rank}[H \,\vdots\, GH \,\vdots\, G^2H] = 2$

* Inorder to calculate $\text{Rank}[H|GH]$ calculate $|H \,\vdots\, GH|$ If $|H \,\vdots\, GH| \neq 0$ Rank=2

* **Output Controllability** :

$$x(K+1)T = Gx(KT) + HU(KT)$$

$$y(KT) = Cx(KT)$$

$G = n \times n$ Matrix

$H = n \times 1$ Matrix

$C = m \times n$ Matrix

$$\rightarrow \boxed{\text{Rank}[CH \,\vdots\, CGH \,\vdots\, CG^2H \,\vdots\, --- \,\vdots\, CG^{n-1}H] = m}$$

* Complete state controllability in pulse Transfer Function:

No Cancellation occur in the pulse-transfer-function. If the Cancellation occurs, the system cannot be controlled

Ex: $$\frac{Y(Z)}{U(Z)} = \frac{\cancel{(Z+0.2)}}{(Z+0.8)\cancel{(Z+0.2)}}$$

so given system is Uncontrollable.

pbl) $$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ -0.8 \end{bmatrix} x(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Sol $$H = \begin{bmatrix} 1 \\ -0.8 \end{bmatrix}, \quad GH = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.8 \end{bmatrix} = \begin{bmatrix} -0.8 \\ -0.16+0.8 \end{bmatrix} = \begin{bmatrix} -0.8 \\ 0.64 \end{bmatrix}$$

$$\text{Rank}\left[H \vdots GH\right] = \text{Rank}\begin{bmatrix} 1 & -0.8 \\ -0.8 & 0.64 \end{bmatrix}$$

$$\begin{vmatrix} 1 & -0.8 \\ -0.8 & 0.64 \end{vmatrix} = 0.64(1) - (-0.8 * -0.8) = 0.64 - 0.64 = 0$$

so Rank $\neq 2$

so given system is Uncontrollable.

* Observability:

The system is said to be completely observable if every initial state $x(0)$ can be determined from the observation of $y(kT)$ over a finite number of sampling periods.

It is useful in Reconstructing unmeasurable state variables

$$x(k+1)T = Gx(kT) + Hu(kT)$$

$$y(kT) = Cx(kT) + Du(kT)$$

$$\Rightarrow \boxed{\text{Rank of}\left[C^T \vdots G^TC^T \vdots \cdots (G^T)^{n-1}C^T\right] = n}$$

* State Observability in pulse-transfer Function:

The pulse-transfer function has no cancellation if and only if the system is completely observable.

82 pbi) $G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$, $H = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 5 & 1 \end{bmatrix}$

Sol

$$C^T = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

$$G^T C^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0-6 \\ 4+0-11 \\ 0+5-6 \end{bmatrix} = \begin{bmatrix} -6 \\ -7 \\ -1 \end{bmatrix}$$

$$(G^T)^2 C^T = (G^T C^T) G^T = \begin{bmatrix} -6 \\ -7 \\ -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$$

$$\text{Rank}\left[C^T \,\vdots\, G^T C^T \,\vdots\, (G^T)^2 C^T \right] = \text{Rank} \begin{bmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & 1 \end{vmatrix} = 4(-7+5) - (-6)[5-5] + 6(-5+7)$$

$$= 4(-2) + 6(0) + 6(2)$$

$$= -8 + 12 = 4 \neq 0$$

Observable

83 # * Principle of Duality b/w Controllability & Observability:

System S_1
$$\begin{cases} x(K+1)T = Gx(KT) + Hu(KT) \\ y(KT) = Cx(KT) \\ G = n\times n \text{ Matrix} \\ H = n\times r \text{ Matrix} \\ C = m\times n \text{ Matrix} \end{cases}$$

System S_2
$$\begin{cases} \hat{x}(K+1)T = G^*\hat{x}(KT) + C^*\hat{u}(KT) \\ \hat{y}(KT) = H^*\hat{x}(KT) \end{cases}$$

G^* = conjugate transpose of 'G'

H^* = conjugate-transpose of 'H'

C^* = conjugate " " 'C'

For system "S_1"

1. Controllability = Rank $[H \vdots GH \vdots G^2H \vdots \cdots \vdots G^{n-1}H] = n$
2. Observability = Rank $[C^T \vdots G^TC^T \vdots \cdots \vdots (G^T)^{n-1}C^T] = n$

For system "S_2"

1. Controllability = Rank $[C^T \vdots G^TC^T \vdots \cdots \vdots (G^T)^{n-1}C^T] = n$
2. Observability = Rank $[H \vdots GH \vdots G^2H \vdots \cdots \vdots G^{n-1}H] = n$

UNIT-IV

* STABILITY ANALYSIS *

One of the most important requirements in the performance of control systems is stability. This is true for continuous systems as well as discrete data & digital systems.

Definitions on stability:

i) Zero state Response (ZSR): The o/p response of a discrete data system that is due to the input only is called ZSR. all the initial conditions of the system are set to zero.

ii) Zero input Response (ZIR): The o/p Response of a discrete data system that is due to the initial conditions only is called ZIR. all the inputs of the system are set to zero.

Total Response = ZSR + ZIR

iii) Bounded-Input Bounded o/p stability (BIBO): For any bounded i/p, the output c is also bounded, then it is said to be BIBO stable.

iv) Bounded Input-Bounded state stability: For any bounded input $u(k)$, the state $x(k)$ is called bounded, then it is said to be BIBS stable.

* Mapping between S-plane and z-plane:

1. Absolute stability and relative stability of linear time invar-iant continuous-time closed loop control system are determined by locations of closed loop poles in the s-plane

2. Complex variables z and s are related by $z = e^{Ts}$

3. The pole and zero locations in the z-plane are related to the pole and zero locations in the s-plane.

4. The dynamic behaviour of the discrete-time control system depends on sampling period T'.

5. A change in the sampling period "T", modifies the poles & zero locations in the z-plane.

*Mapping of the left half of the s-plane into the z-plane:

$$z = e^{ST}$$

where $s = \sigma + j\omega$

$$z = e^{(\sigma + j\omega)T} = e^{\sigma T} \cdot e^{\omega T j}$$

σ related to real axis

$j\omega$ related to Imaginary axis in s-plane

Imaginary axis means Real part is zero

i.e, $e^{\sigma T} = e^{0} = 1$

so Magnitude of z $|z| = 1$

i.e, Imaginary axis in the s-plane ($\sigma = 0$) corresponds to the unit circle in the z-plane, and the interior of the unit circle corresponds to the left half of the s plane.

σ is -ve in the left half of the s-plane.

$$|z| < 1$$

Exterior of the unit circle corresponds to the Right half of the s-plane $|z| > 1$

- LH of s-plane — Inside the unit circle
- RH of s-plane — Outside the "
- Imaginary axis — on the unit circle.

* Primary strips & Complementary strips :

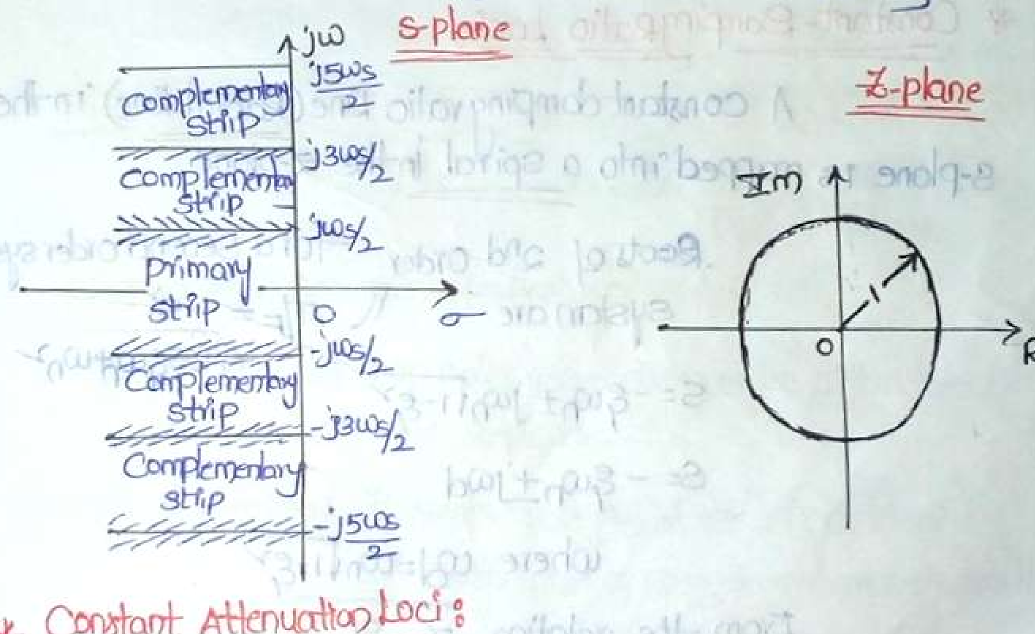
Note that $\angle z = \omega T$ the angle of z varies ~~fro~~ from $-\infty$ to ∞ as ω varies from $-\infty$ to ∞.

1. As point moves from $-\frac{1}{2}j\omega_s$ to $\frac{1}{2}j\omega_s$ on the $j\omega$ axis,
   where ω_s = sampling frequency.
   we have $|z| = 1$ and $\angle z$ varies from $-\pi$ to π in the counterclockwise direction.

2. As point moves from $\frac{1}{2}j\omega_s$ to $j\frac{3}{2}\omega_s$ on the $j\omega$ axis (s plane) the corresponding point in the z-plane trace out the unit circle once in the counter clockwise direction.

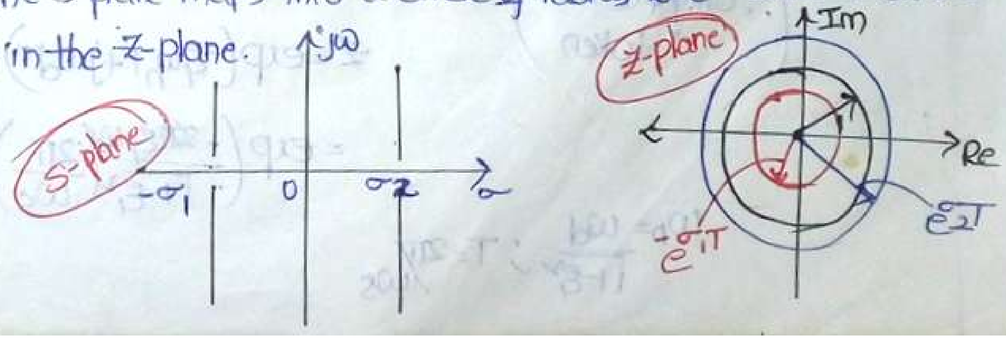
3. As point moves from $-\infty$ to ∞
   Infinite number of times we trace the unit circle.

Left half of the s-plane divided into different strips
1. primary strip $[-\frac{1}{2}j\omega_s$ to $+\frac{1}{2}j\omega_s]$
2. Complementary strips $[\frac{1}{2}j\omega_s$ to $\frac{3}{2}j\omega_s]$



* Constant Attenuation Loci :

A Constant Attenuation Line (a line plotted as σ = constant) in the s-plane maps into a circle of radius $z = e^{\sigma T}$ centered at the origin in the z-plane.

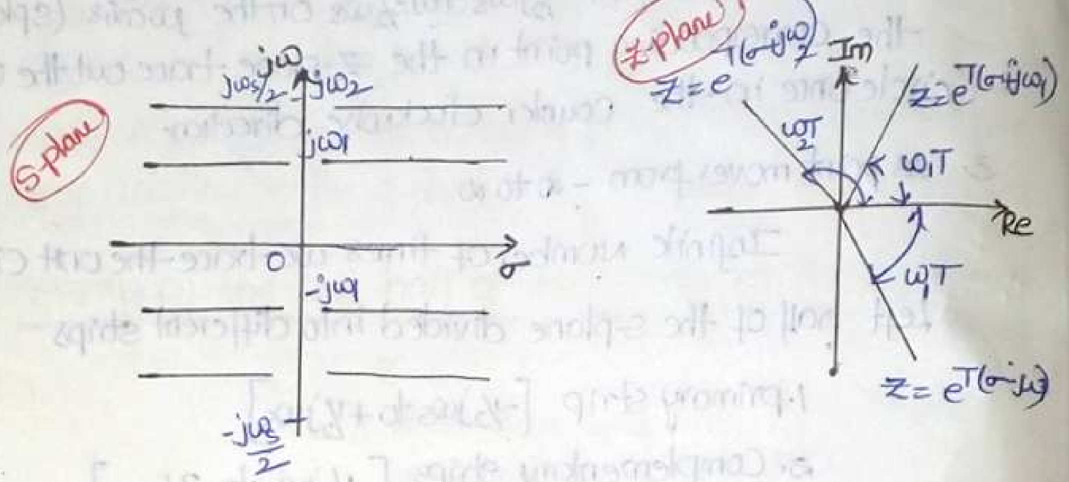


"$-\sigma_1$" line (LHS) in s-plane is mapped into circle with radius $e^{-\sigma T_1}$ (Inside the unit circle) in z-plane.

'σ_2' line (RHS) in s-plane is mapped into circle with radius $e^{\sigma T_2}$ (outside the unit circle) in z-plane.

* Constant Frequency Loci:—

A constant-frequency locus $\omega = \omega_1$ in the s-plane is mapped into a radial line of constant angle $T\omega_1$ in the z-plane.



* Constant-Damping Ratio Loci:

A constant damping ratio line (Radial line) in the s-plane is mapped into a spiral in the z-plane.

Roots of 2nd order system are

for a second order system

$$T/F = \frac{s^2}{s^2+2s\xi\omega_n+\omega_n^2}$$

$$s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$$s = -\xi\omega_n \pm j\omega_d$$

where $\omega_d = \omega_n\sqrt{1-\xi^2}$

From the relation $z = e^{sT}$

(∴ Only +ve root taken)

$$z = \exp(-\xi\omega_n + j\omega_d)T$$

$$z = \exp(-\xi\omega_n T + j\omega_d T)$$

$$= \exp\left(\frac{-2\pi\xi_1\omega_d}{\sqrt{1-\xi_1^2}}\cdot\frac{2\pi}{\omega_s}\right) + j\left(\frac{\omega_d 2\pi}{\omega_s}\right)$$

$$\omega_n = \frac{\omega_d}{\sqrt{1-\xi_1^2}} \; ; \; T = 2\pi/\omega_s$$

$$Z = \exp\left(\frac{-2\pi\xi}{\sqrt{1-\xi^2}}\frac{\omega_d}{\omega_s} + j2\pi\frac{\omega_d}{\omega_s}\right)$$

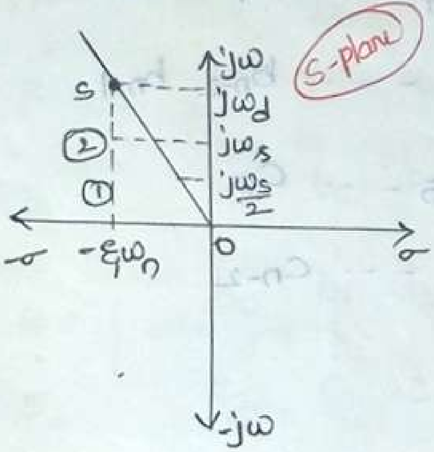
$$|Z| = \exp\left(\frac{-2\pi\xi}{\sqrt{1-\xi^2}}\frac{\omega_d}{\omega_s}\right) \longrightarrow (1)$$

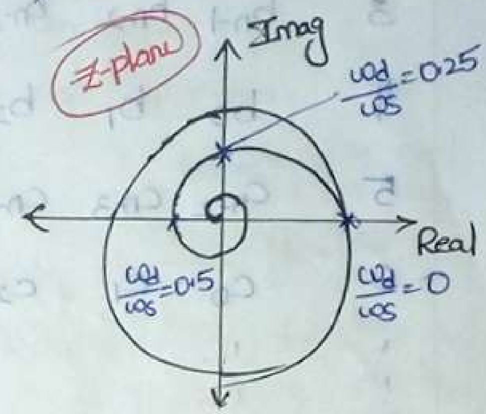
$$\angle Z = 2\pi\left(\omega_d/\omega_s\right) \longrightarrow (2)$$

Consider $\xi = 0.3$

1) $\frac{\omega_d}{\omega_s} = 0.25$ From (1) & (2) $|Z| = 0.6$, $\angle Z = 90^\circ$

2) $\frac{\omega_d}{\omega_s} = 0.5$ From (1) & (2) $|Z| = 0.3$, $\angle Z = 180^\circ$





* Stability Analysis of closed Loop system in the Z-plane:

Consider transfer function $\frac{C(Z)}{R(Z)} = \frac{G(Z)}{1+GH(Z)}$

Characteristic equation $1+GH(Z) = 0$

1. Stable: closed loop poles (or) roots must lie within the unit circle in the Z-plane.

2. Critically stable: If simple pole lies at $Z=1$ (on unit circle)
   If single pair of conjugate complex poles lies on the unit circle.

3. Unstable: Multiple poles on unit circle
   poles outside the unit circle

Jury's stability criterion :

Let us Consider Characteristic equation $P(z)=0$

$$P(z)=a_0z^n+a_1z^{n-1}+a_2z^{n-2}+\dots+a_{n-1}z+a_n=0$$

where $a_0>0$

Row	z^0	z^1	z^2	z^3	----	z^{n-2}	z^{n-1}	z^n
1	a_n	a_{n-1}	a_{n-2}	a_{n-3}	---	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3	---	a_{n-2}	a_{n-1}	a_1
3	b_{n-1}	b_{n-2}	b_{n-3}	b_{n-4}	----	b_1	b_0	
4	b_0	b_1	b_2	b_3	---	b_{n-2}	b_{n-1}	
5	c_{n-2}	c_{n-3}	c_{n-4}	c_{n-5}	---	c_0		
6	c_0	c_1	c_2	c_3	---	c_{n-2}		
⋮	⋮							
2n-5	P_3	P_2	P_1	P_0				
2n-4	P_0	P_1	P_2	P_3				
2n-3	q_2	q_1	q_0					

No. of rows $2n-3$ where "n" = order of Characteristic Equation.

Stability criterion by the Jury Test:

$$P(z)=a_0z^n+a_1z^{n-1}+\dots+a_{n-1}z+a_n$$

1) $a_0>0$
2) $|a_n|<a_0$
3) $P(z)\big|_{z=1}>0$
4) $P(z)\big|_{z=-1}\begin{cases}<0 & \text{for "n" odd}\\ >0 & \text{for "n" even}\end{cases}$
5) $|b_{n-1}|>|b_0|$

   $|c_{n-2}|>|c_0|$

   ⋮

   $|q_2|>|q_0|$

If all conditions are satisfied given system is stable

91 1) $P(z) = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$, where $a_0 > 0$

Sol 1. $|a_4| < a_0$

2. $P(1) = a_0 + a_1 + a_2 + a_3 + a_4 > 0$

3. $P(-1) = a_0 - a_1 + a_2 - a_3 + a_4 > 0$ $n = 4 = $ Even

4. $|b_3| > |b_0|$

$|c_2| > |c_0|$

No. of Rows $= 2n-3 = 2(4)-3 = 5$

Rows	z^0	z^1	z^2	z^3	z^4
1	a_4	a_3	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3	a_4
3	b_3	b_2	b_1	b_0	
4	b_0	b_1	b_2	b_3	
5	c_2	c_1	c_0		

Where $b_3 = \begin{vmatrix} a_4 & a_0 \\ a_0 & a_4 \end{vmatrix}$

$b_2 = \begin{vmatrix} a_4 & a_1 \\ a_0 & a_3 \end{vmatrix}$

$b_1 = \begin{vmatrix} a_4 & a_2 \\ a_0 & a_2 \end{vmatrix}$

$b_0 = \begin{vmatrix} a_4 & a_3 \\ a_0 & a_1 \end{vmatrix}$

Where $c_2 = \begin{vmatrix} b_3 & b_0 \\ b_0 & b_3 \end{vmatrix}$; $c_1 = \begin{vmatrix} b_3 & b_1 \\ b_0 & b_2 \end{vmatrix}$; $c_0 = \begin{vmatrix} b_3 & b_2 \\ b_0 & b_1 \end{vmatrix}$

$c_2 > c_0 \longrightarrow$ system is stable.

2) $P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08 = 0$ check stability by using Jury's stability.

Sol $P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08 = 0$

$a_0 = 1$; $a_1 = -1.2$, $a_2 = 0.07$, $a_3 = 0.3$, $a_4 = -0.08$

1. $|a_4| < a_0$, $|-0.08| < 1$, $0.08 < 1$

2. $P(1) = 1 - 1.2 + 0.07 + 0.3 - 0.08 = 0.09 > 0$.

3. $P(-1) = 1 + 1.2 + 0.07 - 0.3 - 0.08 = 1.89 > 0$ ($n = 4$ Even)

4. $|c_2| > |c_0|$

Row	z^0	z^1	z^2	z^3	z^4
1	$\overset{a_4}{-0.08}$	$\overset{a_3}{0.3}$	$\overset{a_2}{0.07}$	$\overset{a_1}{-1.2}$	$\overset{a_0}{1}$
2	$\overset{a_0}{1}$	$\overset{a_1}{-1.2}$	$\overset{a_2}{0.07}$	$\overset{a_3}{0.3}$	$\overset{a_4}{-0.08}$
3	$\overset{b_3}{-0.994}$	$\overset{b_2}{1.176}$	$\overset{b_1}{-0.0756}$	$\overset{b_0}{-0.204}$	
4	$\overset{b_0}{-0.204}$	$\overset{b_1}{-0.075}$	$\overset{b_2}{1.176}$	$\overset{b_3}{-0.994}$	
5	$\overset{c_2}{0.946}$	$\overset{c_1}{-1.184}$	$\overset{c_0}{0.315}$		

Where $b_3 = \begin{vmatrix} -0.08 & 1 \\ 1 & -0.08 \end{vmatrix}$; $b_2 = \begin{vmatrix} -0.08 & -1.2 \\ 1 & 0.3 \end{vmatrix}$; $b_1 = \begin{vmatrix} -0.08 & 0.07 \\ 1 & 0.07 \end{vmatrix}$

$= -0.994$ $\quad = 1.176 \quad = -0.0756$

$b_0 = \begin{vmatrix} -0.08 & 0.3 \\ 1 & -1.2 \end{vmatrix} = -0.204$

similarly

$c_2 = \begin{vmatrix} -0.994 & -0.204 \\ -0.204 & -0.994 \end{vmatrix}$; $c_1 = \begin{vmatrix} -0.994 & -0.0756 \\ -0.204 & 1.176 \end{vmatrix}$; $c_0 = \begin{vmatrix} -0.994 & 1.176 \\ -0.204 & -0.0[illegible] \end{vmatrix}$

$= 0.946 \qquad = -1.184 \qquad = 0.315$

5. $|c_2| > |c_0|$

$|0.946| > |0.315|$ ✓

all conditions satisfied so system is stable (Given $p(z)$)

3) $p(z) = z^3 + 3.3z^2 + 4z + 0.8 = 0$

$a_0 = 1, \; a_1 = 3.3, \; a_2 = 4, \; a_3 = 0.8$

No of rows $= 2n-3 = 2(3)-3$
$= 3$

1. $|a_0| > 0$ ✓
2. $|a_3| < a_0, \; |0.8| < 1$ ✓
3. $P(1) = 1 + 3.3 + 4 + 0.8 = 9.1 > 0$ ✓
4) $P(-1) = -1 + 3.3 - 4 + 0.8 = 4.1 - 5 = -0.9 < 0$ ✓ (n odd)

Row	z^0	z^1	z^2	z^3
1	0.8	4	3.3	1
2	1	3.3	4	0.8
3	$\begin{vmatrix} 0.8 & 1 \\ 1 & 0.8 \end{vmatrix} = -0.36$	$\begin{vmatrix} 0.8 & 3.3 \\ 1 & 4 \end{vmatrix} = -0.1$	$\begin{vmatrix} 0.8 & 4 \\ 1 & 3.3 \end{vmatrix} = -1.36$	

$|b_2| > |b_0|$

$|-0.36| > |-1.36|$ Not satisfied.

So system is Unstable.

4) $z^3 - 1.1z^2 - 0.1z + 0.4 = 0$

5) $z^3 + 4.4z^2 + 5z + 0.8 = 0$

6) $5z^2 - 2z + 2 = 0$

7) $z^3 - 0.2z^2 - 0.25z + 0.05 = 0$

8) $z^4 - 1.7z^3 + 1.04z^2 - 0.268z + 0.024 = 0$

9) $z^3 + kz^2 + 1.5kz - (k+1) = 0$ system stable - find the Range of 'k'.

6) $5z^2 - 2z + 2 = 0$

Given $a_0 = 5$, $a_1 = -2$, $a_2 = 2$ $n = 2$ [heighest order of 'z']

No. of rows $= 2n - 3 = 2(2) - 3 = 4 - 3 = 1$

1. $|a_0| > 0$, $|5| > 0$ ✓
2. $|a_2| < a_0$ $|2| < 5$ ✓
3. $p(1) = 5(1)^2 - 2(1) + 2 = 5 - 2 + 2 = 5 > 0$ ✓
4. $p(-1) = 5(-1)^2 - 2(-1) + 2 = 5 + 2 + 2 = 9 > 0$ [for n even] ✓

Row	z^0	z^1	z^2
1	2	-2	5

$|b_2| > |b_0| \Rightarrow |2| > |5|$

Not satisfied

Unstable.

9) $p(z) = z^3 + Kz^2 + 1.5Kz - (K+1)$ — find the Range of 'K' If the given system is stable.

Sol $p(z) = z^3 + Kz^2 + 1.5Kz - (K+1)$

$a_0 = 1,\ a_1 = K,\ a_2 = 1.5K,\ a_3 = -(K+1)$

1. $|a_0| > 0,\ |1| > 0$ ✓

2. $|a_3| < a_0,\ |-(K+1)| < 1$

$K+1 < 1$

$K < 1-1$

$K < 0$ ✓

3. $p(1) > 0$

$p(1) = (1)^3 + K(1)^2 + 1.5K(1) - (K+1) > 0$

$1 + \cancel{K} + 1.5K - \cancel{K} - 1 > 0$

$1.5K > 0$

95 10) Consider the discrete unity feedback control system (whose sampling period $T = 1\ sec$) whose openloop pulse transfer function is given by

$$G(z) = \frac{K(0.3679z+0.2642)}{(z-0.3679)(z-1)}$$ Determine the range of "K" for stability

By use of Jury stability test.

sol Given $G(z) = \frac{K(0.3679z+0.2642)}{(z-0.3679)(z-1)}$; $H(z)=1$

Transfer Function $\frac{G(z)}{1+G(z)H(z)} = \frac{C(z)}{R(z)} = \frac{K(0.3679z+0.2642)}{z^2+(0.3679z-1.3679)z+0.3679+0.2642K}$

Thus the characteristic equation

$$P(z) = z^2+(0.3679K-1.3679)z+0.3679+0.2642K=0 \rightarrow (A)$$

since this is a second-order system, the Jury stability conditions may be written as follows:

$a_0 = 1$, $a_1 = 0.3679K-1.3679$

$a_2 = 0.3679+0.2642K$

1. $|a_2| < a_0$

$|0.3679+0.2642K| < 1$

$0.2642K < 1-0.3679$

$0.2642K < 0.6321$

$K < 2.39251 \rightarrow (1)$

2. $P(1) > 0$

$P(1) = 1+(0.3679K-1.3679)+0.3679+0.2642K > 0$

$0.6321K > 0$

$K > 0 \rightarrow (2)$

3. $P(-1) = 1-(0.3679K-1.3679)+0.3679+0.2642K$ $\quad n=2$ (even) $P(-1)>0$

$2.7358 - 0.1037K > 0$

$26.382 > K \rightarrow (3)$

from above (3) consider $\underline{K = 2.3925}$

Substitute 'K' value in equation (A) then we get

$$P(z) = z^2+[0.3679(2.3925)-1.3679]z+0.3679+0.2642(2.3925)$$

$$P(z) = z^2-0.4877z+1=0 \rightarrow (4)$$

Calculate roots of equation (4) $z = 0.2439 \pm j0.9698$

Frequency of sustained oscillations

$$\omega_d = \frac{\omega_s}{2\pi}\angle z = \frac{2\pi}{2\pi}\angle z = \tan^{-1}\left(\frac{0.9698}{0.2439}\right) = \underline{1.3244\ rad/sec}$$

$= 75°$

π radians $= 180°$ $\rightarrow$ 1 radian $= \frac{180°}{\pi} = \frac{0.005}{\pi}$

* Stability Analysis by use of the bilinear transformation & Routh stability criterion :-

This method requires transformation from the z-plane to another complex plane, the ω-plane.

The bilinear transformation defined by

$$z = \frac{\omega+1}{\omega-1}$$

from the above : $\omega = \frac{z+1}{z-1}$

maps the inside of the unit circle in the z-plane into the left half of the ω-plane.

Let the real part of ω be called 'σ' & the imaginary part ω, so that

$$\omega = \sigma + j\omega$$

since the inside of the unit circle in the z-plane is

$$|z| = \left|\frac{\omega+1}{\omega-1}\right| = \left|\frac{\sigma+j\omega+1}{\sigma+j\omega-1}\right| < 1$$

[or] $$\frac{(\sigma+1)^2+\omega^2}{(\sigma-1)^2+\omega^2} < 1$$

$$(\sigma+1)^2+\omega^2 < (\sigma-1)^2+\omega^2$$

$$\cancel{\sigma^2}+\cancel{1}+2\sigma+\cancel{\omega^2} < \cancel{\sigma^2}+\cancel{1}-2\sigma+\cancel{\omega^2}$$

$$4\sigma < 0$$

$$\underline{\underline{\sigma < 0}}$$

Thus, the inside of the unit circle in the z-plane ($|z|<1$) corresponds to the left half of the ω-plane.

97 pb1) Consider the characteristic Equation,

$$P(z) = z^2 - 0.25 = (z-0.5)(z+0.25) = 0$$

sol

$$z = \frac{w+1}{w-1}$$

$$P(z) = \left(\frac{w+1}{w-1}\right)^2 - 0.25 = 0$$

$$(w+1)^2 - 0.25(w-1)^2 = 0$$

$$w^2 + 1 + 2w - 0.25(w^2 + 1 - 2w) = 0$$

$$0.75w^2 + 2.5w + 0.75 = 0$$

Now apply the Routh's stability criterion.

$a_0 = 0.75$, $a_1 = 2.5$, $a_2 = 0.75$ order $n=2$

w^2	0.75	0.75
w^1	2.5	0
w^0	$\frac{2.5 * 0.75 - 0.75(0)}{2.5} = 0.75$	

Hence all the coefficients are of the same sign & All the roots of the CE are lies in the left half of the s-plane & the system is stable.

pb2) $P(z) = z^3 - 1.25z^2 - 1.375z - 0.25 = 0$

P(z) into Q(w)=0

$$z = \frac{w+1}{w-1}$$

$$\left(\frac{w+1}{w-1}\right)^3 - 1.25\left(\frac{w+1}{w-1}\right)^2 - 1.375\left(\frac{w+1}{w-1}\right) - 0.25 = 0$$

$$(w+1)^3 - 1.25(w+1)^2(w-1) - 1.375(w+1)(w-1)^2 - 0.25(w-1)^3 = 0$$

$$-1.875w^3 + 3.875w^2 + 4.875w + 1.125 = 0$$

Applying Routh's stability criterion

−	w^3	−1.875	4.875
+	w^2	3.875	1.125
	w^1	5.419	0
	w^0	1.125	

one sign change in first column [So one pole on RHS so system is unstable.